ON-LINE CHAIN PARTITIONING OF UP-GROWING ORDERS: THE CASE OF 2-DIMENSIONAL ORDERS AND SEMI-ORDERS

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Abstract. We analyze special cases of the on-line chain partition problem of up-growing orders. One result is a lower bound for 2-dimensional orders. Together with the old upper bound this shows that the precise value of this game is $\binom{w+1}{2}$, where w is the width of the order. Our main contribution is the analysis of the game for semi-orders. Surprisingly the golden ratio comes into play, the precise value of the game for width w is $\lfloor \frac{1+\sqrt{5}}{2} w \rfloor$. The proof of the upper bound is based on a quite unusual twist in perspective. It is established by showing that against a natural algorithm the best strategy is the strategy provided by the lower bound construction.

1. Introduction

On-line chain partitioning of a poset can be viewed as a two-person game between Algorithm and Spoiler. The game is played in rounds. Spoiler presents an on-line order, one point at a time. Algorithm responds by making an irrevocable assignment of the new point to one of the chains of the chain partition. (Throughout this paper we often treat chains as colors and write 'p got color i' instead of 'p was assigned to chain i'). The performance of Algorithm's strategy is measured by comparing the number of chains used with the number of chains of an optimal chain partition. By Dilworth's Theorem the size of an optimal chain partition equals the width of the order. The value of the game for orders of width w, denoted further by $\mathsf{val}(w)$, is the largest integer n for which Spoiler has a strategy that forces any Algorithm to use n chains. Alternatively, it is the least integer n such that some Algorithm has a strategy using at most n chains for every feasible on-line order of width w.

The study of chain partitioning games goes back to the early 80's when Kierstead [4] (upper bound) and Szemerédi (lower bound published in [5]) proved the so far best estimates for on-line orders of width w:

$$\binom{w+1}{2} \leqslant \mathsf{val}(w) \leqslant \frac{5^w - 1}{4}.$$

The precise values are only known for $w \leqslant 2$. Kierstead [4] showed that 5 is a lower and 6 an upper bound in the case of orders of width two. Felsner [3] closed the gap by reducing the upper bound to 5. On the restricted class of on-line interval orders Kierstead and Trotter [7] obtained a precise result: $\mathsf{val}(w) = 3w - 2$ for on-line interval orders.

Up-growing on-line orders have been introduced by Felsner [3]. In this variant Spoiler's power is restricted by the condition that the new element has to be a maximal element of the order presented so far. Felsner [3] showed that the value of the chain partition game on up-growing orders is $\binom{w+1}{2}$. The case of up-growing interval orders was recently resolved by Baier, Bosek and Micek [1], they proved $\mathsf{val}(w) = 2w - 1$.

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This paper continues the investigation of on-line chain partitioning for special classes of up-growing orders. Specifically, it deals with 2-dimensional orders and with semi-orders.

An order $\mathcal{P} = (X, \leqslant)$ is *2-dimensional* iff it has a *2-realizer*, i.e., a pair of linear extensions, $L_1 = (X, <_1)$ and $L_2 = (X, <_2)$ such that for all $x, y \in X$ we have $x < y$ iff $x <_1 y$ and $x <_2 y$. An order $\mathcal{P}$ is called a *semi-order* if it has a unit interval representation, i.e., there exists a mapping I of points of the order into unit length intervals on a real line so that $x < y$ in $\mathcal{P}$ iff interval $I(x)$ is entirely to the left of $I(y)$. Trotter's book [11] is a comprehensive account to order theory. Möhring [8] gives a valuable overview on representations of 2-dimensional orders.

The value of the on-line chain partitioning game of 2-dimensional orders is known to be $\binom{w+1}{2}$. The upper bound comes from the upper bound for on-line antichain partitions due to Schmerl [9]. The lower bound was proved by Szemerédi [10], this lower bound still holds in the more restricted version of the game where Spoiler provides a realizer of the presented order (in other words he presents points in the plane). In Section 2 we show that the same value holds even in the more restricted version where a 2-dimensional up-growing order is presented with a realizer.

Considering on-line chain partitions of semi-orders note that the general (not up-growing) case is easy to analyze: First, observe that the number of colors used by Algorithm can be bounded by $2w - 1$. To see this consider the set $\mathsf{Inc}\,(x)$ of elements incomparable to the element x which has to be colored. Clearly $\mathsf{width}\,(\mathsf{Inc}\,(x)) \leqslant w - 1$ since the width of the whole order does not exceed w. Moreover, $\mathsf{height}\,(\mathsf{Inc}\,(x)) \leqslant 2$ as the presented order is $(\mathbf{3}+\mathbf{1})$-free (see Theorem 3.1). This implies that $|\mathsf{Inc}\,(x)\,| \leqslant 2(w-1) = 2w - 2$, proving that at least one of $2w - 1$ colors is legal for x.

It turns out that there is no better strategy for Algorithm. In other words, Spoiler may force Algorithm to use $2w - 1$ chains on semi-orders of width w. A strategy for Spoiler looks as follows:

(1) Present two antichains A and B, both consisting of w points in such a way that $A < B$, i.e., all points from A are below all points from B. If Algorithm uses $2w - 1$ or more colors, the construction is finished. Otherwise, suppose that k colors $c_1, \dots, c_k$ $(2 \leqslant k \leqslant w)$ are used twice, once in A and once in B.

(2) Present $k-1$ points $x_1, \dots, x_{k-1}$ in such a way that the interval representation of the whole order looks as in Figure 1. It is easy to verify that in such setting Algorithm is forced to use $2w - 1$ chains.

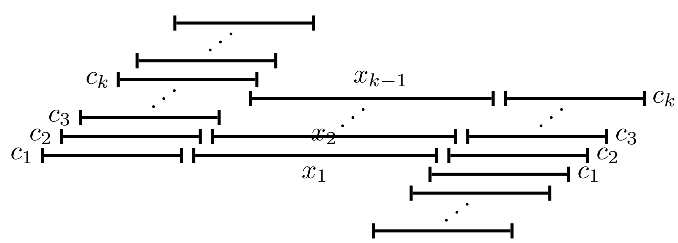


FIGURE 1. Strategy for Spoiler forcing Algorithm to use $2w - 1$ chains

In Section 3 we deal with the up-growing variant of the problem for semi-orders. We prove matching upper and lower bounds of $\lfloor \frac{1+\sqrt{5}}{2} w \rfloor$ in this case. The last section contains some open problems in the area.

1.1. **Notation.** An *on-line poset* is a poset $\mathcal{P} = (P, \leqslant)$ with an additional ordering of the elements of P representing the order in which points of $\mathcal{P}$ are introduced. To emphasize the on-line nature we often write P as a sequence $P = (p_1, \dots, p_n)$.

An on-line order $\mathcal{P} = (P, \leqslant)$ with $P = (p_1, \ldots, p_n)$ is called *up-growing* if the order of the sequence $(p_1, \ldots, p_n)$ is a linear extension of $\mathcal{P}$, i.e., $p_i < p_j$ implies $i < j$. This is equivalent to the restriction that for every i the element p_i introduced in the ith round is a maximal element of the order $\mathcal{P}_i = ((p_1, \ldots, p_{i-1}, p_i), \leqslant)$, i.e., in the restriction of $\mathcal{P}$ to the first i elements.

Let $x{\downarrow_{\mathcal{P}}} = \{y \in P : y < x\}$, called a *downset* of x in $\mathcal{P}$, denote the set of predecessors of x in $\mathcal{P}$. Dually, let $x{\uparrow_{\mathcal{P}}} = \{y \in P : y > x\}$, called an *upset* of x, denote the set of successors of x in $\mathcal{P}$. If the order $\mathcal{P}$ is unambiguous from the context we also write $x{\uparrow}$ instead of $x{\uparrow_{\mathcal{P}}}$ and $x{\downarrow}$ instead of $x{\downarrow_{\mathcal{P}}}$. The maximum element of a chain γ is denoted by $\mathsf{top}(\gamma)$.

2. Bounds for 2-dimensional orders

We have been motivated to study lower bounds for the value of the 2-dimensional up-growing on-line chain partitioning game by the following two results:

Theorem 2.1 (Szemerédi)**.** *The value of the on-line chain partitioning game on the class of 2-dimensional on-line orders of width at most w is at least $\binom{w+1}{2}$. This remains true if the presentation of the on-line order is done by presenting a 2-realizer.*

The on-line order constructed in the proof of this theorem as given in [5] is not up-growing.

Theorem 2.2 (Felsner)**.** *The value of the on-line chain partitioning game on the class of up-growing on-line orders of width at most w is at least $\binom{w+1}{2}$.*

The up-growing order which was used in [3] to prove this lower bound is not 2-dimensional. Here we show that the value $\binom{w+1}{2}$ remains a lower bound even if we consider on-line orders which are both: up-growing and 2-dimensional. Together with the upper bound for the up-growing on-line chain partitioning game on unrestricted orders from [3] we obtain:

Theorem 2.3. *The value of the on-line chain partitioning game on the class of up-growing 2-dimensional orders of width at most w is exactly $\binom{w+1}{2}$. This remains true even when the presentation of the on-line order is done by presenting a 2-realizer.*

The proof for the lower bound is inspired by the proof of Theorem 2.2. However, we have to take care that all construction steps preserve the dimension. This is achieved by restricting the operations used by the strategy to only very elementary ones. For the description of the operations we need an easy fact about 2-dimensional orders.

Claim 2.4. If $\mathcal{P} = (P, \leqslant)$ is 2-dimensional with 2-realizer L_1, L_2 and the maximal elements of P are ordered as in L_1, i.e., $\mathsf{Max}(P) = \{x_1, \ldots, x_w\}$ and $x_1 <_1 x_2 <_1 \ldots <_1 x_w$, then their order is reversed in L_2, i.e., $x_w <_2 \ldots <_2 x_2 <_2 x_1$. We call $(x_1, \ldots, x_w)$ the *sorted antichain of maximal elements* of $\mathcal{P}$.

Given the sorted antichain $(x_1, \ldots, x_w)$ of maxima and two indices $1 \leqslant i \leqslant j \leqslant w$ we introduce the following operations:

$\mathsf{above}_{i,j}$ Add a new element y with relations $x_i < y$, $x_{i+1} < y, \ldots, x_j < y$ and all relations implied by transitivity but no others.

$\mathsf{left}_{i,j}$ Add a set $y_{i+1}, y_{i+2}, \ldots, y_j$ of twin elements such that each y_s from this set has relations $x_{i+1} < y_s, \ldots, x_j < y_s$ and all relations implied by transitivity but no others. In this case the index of the element y introduced by the preceding move $\mathsf{above}_{i,j}$ is i, i.e., $y_i = y$.

$\mathsf{right}_{i,j}$ Add a set $y_i, y_{i+1}, \dots, y_{j-1}$ of twin elements such that each y_s from this set has relations $x_i < y_s, \dots, x_{j-1} < y_s$ and all relations implied by transitivity but no others. In this case the index of the element y introduced by the preceding move $\mathsf{above}_{i,j}$ is j, i.e., $y_j = y$.

The combination of a move $\mathsf{above}_{i,j}$ followed by $\mathsf{left}_{i,j}$ is illustrated in Figure 2. Throughout the strategy Spoiler repeatedly makes a move of type $\mathsf{above}_{i,j}$, depending on the color given to the new element y Spoiler completes the operation with a move of type either $\mathsf{left}_{i,j}$ or $\mathsf{right}_{i,j}$.

Claim 2.5. If L_1, L_2 is a 2-realizer of P and P^+ is obtained by a move $\mathsf{above}_{i,j}$ followed by $\mathsf{left}_{i,j}$, then there is a 2-realizer L_1^+, L_2^+ of P^+. The same holds if $\mathsf{above}_{i,j}$ is followed by $\mathsf{right}_{i,j}$. In other words: The operations preserve the 2-dimensionality of the order.

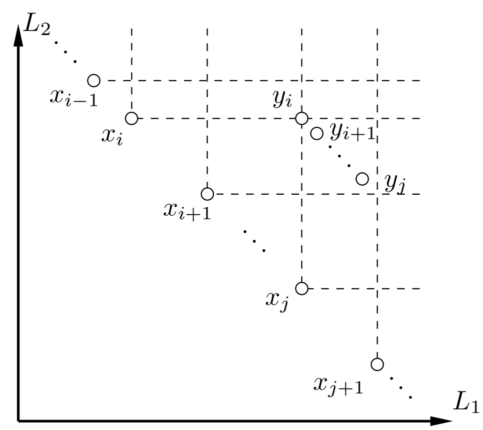


FIGURE 2. Combination of $\mathsf{above}_{i,j}$ followed by $\mathsf{left}_{i,j}$

If x is a maximal element of an order partitioned into chains (colors) then $\mathsf{private}(x)$ is the set of colors γ with $\mathsf{top}(\gamma) \leqslant x$ and $\mathsf{top}(\gamma) \nleqslant y$ for all maximal elements $y \neq x$. (Recall that $\mathsf{top}(\gamma)$ denotes the maximum element of the chain γ).

The work horse for the proof of the theorem is the following proposition.

Proposition 2.6. *Given a number $Z \in \mathbb{N}$. Let P be a 2-dimensional order of width w with sorted antichain $(x_1, \dots, x_w)$ of maximal elements and let a chain partition of P be given. There is a strategy $S(i,j)$, for all i, j, which extends P in an up-growing way by using the above operations such that the width remains w and every on-line chain partitioning algorithm has to tolerate one of the following two results for the sorted antichain of maximal elements $z_1, \dots, z_w$ of the resulting order:*

(1) $|\mathsf{private}(z_r)| \geqslant j - r + 1$ *for all* $r = i, \dots, j$ *or*

(2) *the algorithm has used more than Z colors.*

Moreover for all $s \notin \{i, i+1, \dots, j\}$ we have $z_s = x_s$ and $\mathsf{private}(x_s)$ *was not affected by the play of $S(i,j)$.*

Proof. The proof is by induction on $j - i$. For $j = i$ we are in case (1) without doing anything, just observe that the color of the chain to which x_i has been assigned is an element of $\mathsf{private}(x_i)$, hence $|\mathsf{private}(x_i)| \geqslant 1$.

For the induction step we begin with strategy $S(i, j-1)$ which may result in case (2) so that we can stop. In the interesting case $S(i, j-1)$ ends with a sorted antichain of maximal elements $y_1, \ldots, y_w$ such that $|\mathsf{private}(y_r)| \geqslant (j-1)-r+1 = j-r$ for $r = i, \ldots, j-1$. The next step is a move of type $\mathsf{above}_{i,j}$. Let the new element y be colored by γ, we distinguish two cases:

(a) If $\gamma \notin \mathsf{private}(y_i)$ then a move $\mathsf{left}_{i,j}$ follows. This results in a sorted antichain $(y_1, \ldots, y_w)$ of maximal elements with $|\mathsf{private}(y_i)| \geqslant j-i+1$. Playing $S(i+1, j)$ results in one of the two outcomes claimed for $S(i, j)$.

(b) If $\gamma \in \mathsf{private}(y_i)$ then continue with a move $\mathsf{right}_{i,j}$. This results in a sorted antichain $(y_1, \ldots, y_w)$ of maximal elements with the additional color γ in the set $\mathsf{private}(y_j)$. Continue with another iteration of strategy $S(i, j-1)$. This or one of the following iterations of $S(i, j-1)$ may result in case (a). If case (a) is avoided, then after Z iterations we have $|\mathsf{private}(y_j)| \geqslant Z$ and, hence, state (2) of the proposition.

□

To prove the theorem we fix $Z > \binom{w+1}{2}$. Starting with an initial antichain $x_1, \ldots, x_w$ apply strategy $S(1, w)$. Either we have reached Z colors, or otherwise, the final sorted antichain $z_1, \ldots, z_w$ of maximal elements has the property that the private colors of the elements obey $|\mathsf{private}(z_i)| \geqslant w-i+1$ for each $1 \leqslant i \leqslant w$. Hence, the total number of chains is $\binom{w+1}{2}$. □

3. Up-growing semi-orders

3.1. **Outline.** In Section 3.2 we collect some facts about semi-orders. Section 3.3 deals with *natural algorithms* for coloring up-growing semi-orders. We show that they are optimal algorithms. Sections 3.4–3.6 present the lower bound. For the analysis of the strategy of Spoiler it is of great help to know that we may assume that the coloring algorithm may respond with a natural coloring. Section 3.7 is called the upper bound. What we actually show there is that the strategy from the previous section is an optimal strategy for Spoiler. In a sequence of steps we modify a given strategy for Spoiler to bring it into a form which is very similar to the form of the lower bound strategy. For each modifying step we prove that it does not reduce the number of colors it enforces. Having obtained the normalized form the desired result is obtained by analyzing the parameters of the strategy. Essentially this analysis amounts in solving a linear program whose optimal solution is the golden ratio.

3.2. **Basic facts.** An order $\mathcal{P}$ is called an interval order if it has an interval representation, i.e., there exists a mapping I of points of the order into intervals on a real line so that $x < y$ in $\mathcal{P}$ iff $I(x) < I(y)$. Interval order $\mathcal{P}$ may be characterized by the fact that the set of downsets (upsets) of points in $\mathcal{P}$ is linearly ordered with respect to $\subseteq$, i.e., for any two $p, q \in P$ either $p{\downarrow} \subseteq q{\downarrow}$ or $p{\downarrow} \supseteq q{\downarrow}$ ($p{\uparrow} \subseteq q{\uparrow}$ or $p{\uparrow} \supseteq q{\uparrow}$). (This ordering of downsets (upsets) corresponds to the order of left (right) endpoints in an interval representation). Another characterization of interval orders is in the terms of the $(\mathbf{2}+\mathbf{2})$ from Figure 3: $\mathcal{P}$ is an interval order iff it does not contain a $(\mathbf{2}+\mathbf{2})$ as an induced suborder.

Downsets and upsets of points in $\mathcal{P}$ can be used to define extensions of $\mathcal{P}$ as follows:

$$p \prec_{\downarrow} q \quad \text{iff} \quad p{\downarrow} \subsetneq q{\downarrow} \text{ or } (p{\downarrow} = q{\downarrow} \text{ and } p{\uparrow} \supsetneq q{\uparrow}), \tag{1}$$

$$p \prec_{\uparrow} q \quad \text{iff} \quad q{\uparrow} \subsetneq p{\uparrow} \text{ or } (p{\uparrow} = q{\uparrow} \text{ and } q{\downarrow} \supsetneq p{\downarrow}). \tag{2}$$

In general, two orderings $\prec_{\downarrow}$ and $\prec_{\uparrow}$ need not be the same. Indeed, in a $(\mathbf{3}+\mathbf{1})$ poset from Figure 3 we have $d \prec_{\downarrow} b$ whereas $b \prec_{\uparrow} d$. An interval order $\mathcal{P} = (P, \leqslant)$

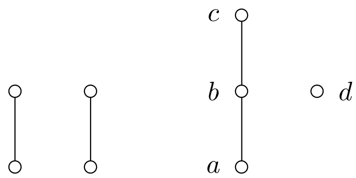


FIGURE 3. $(\mathbf{2}+\mathbf{2})$ and $(\mathbf{3}+\mathbf{1})$ orders

is called a *semi-order* if it has a unit interval representation, i.e., all intervals are of unit length. We restate from [11] the following characterization theorem for semi-orders.

Theorem 3.1 (Scott, Suppes). *Let $\mathcal{P} = (P, \leqslant)$ be an interval order. Then the following statements are equivalent.*

(i) *$\mathcal{P}$ is a semi-order,*
(ii) *$\mathcal{P}$ is a $(\mathbf{3}+\mathbf{1})$-free order, i.e., $\mathcal{P}$ does not contain elements $a, b, c, d \in P$ such that $a < b < c$ and $d \parallel \{a, b, c\}$ (see Figure 3),*
(iii) *The two orderings $\prec_\downarrow$ and $\prec_\uparrow$ are identical in $\mathcal{P}$.*

Let $\mathcal{P}$ be a semi-order and let $p, q \in P$. Theorem 3.1 justifies the following definition

$$p \prec q \quad \text{iff} \quad p \prec_\downarrow q \quad (\text{or } p \prec_\uparrow q).$$

Another way of thinking of the $\prec$ relation is in terms of down-sets or up-sets:

$$p \prec q \quad \text{iff} \quad p{\downarrow} \subsetneq q{\downarrow} \quad \text{or} \quad p{\uparrow} \supsetneq q{\uparrow}. \tag{3}$$

3.3. **Natural algorithms.** We introduce the idea of a natural algorithm for chain partitioning of an up-growing semi-order on the following example. Consider semi-order $\mathcal{P} = (P, \leqslant)$ with $P = (a, b, c, d, e)$ and the coloring $\Gamma : P \setminus \{e\} \to \mathbb{N}$ as shown in Figure 4. Point e may be colored with a *new color* (say, with color 4) or with

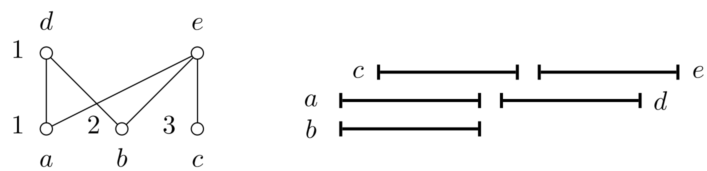


FIGURE 4. Poset $\mathcal{P}$ with its unit interval representation and coloring Γ of points a, b, c, d

one of the colors already used, i.e., with an *old color*. In the latter case Algorithm may choose between colors 2 and 3. (We say that color α is *valid* for the new point p extending an already colored poset $\mathcal{P}$ if p dominates all points colored with α in $\mathcal{P}$). We claim that among the valid colors 2 and 3 defining $\Gamma(e) = 3$ is the better choice. Indeed, any future point p presented by Spoiler dominating c also dominates b (otherwise, $\mathcal{P}$ would have a $(\mathbf{2}+\mathbf{2})$ configuration which is forbidden in semi-orders). On the other hand, Spoiler may play q greater than b but remaining incomparable to c (see Figure 5). Hence, using the color of c for e leaves more coloring options for the future. It turns out that whenever the colors of two points x and y are valid and $x \prec y$, then it is advantageous to use the color of y.

Definition 3.2. Let p be a new point presented by Spoiler and let $\mathsf{validTops}(p)$ denote the set of predecessors of p in $\mathcal{P}$ being the tops of colors valid for p. Point p is *naturally colored* if the following holds:

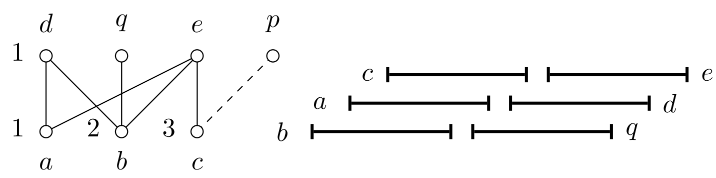


FIGURE 5. Point q may be presented by Spoiler in the future, point p can not.

(1) If $\mathsf{validTops}(p)$ is not empty, then p is assigned the color of some q which is $\prec$–maximal in $\mathsf{validTops}(p)$.

(2) If $\mathsf{validTops}(p)$ is empty, p receives any new color.

A coloring is *natural* if every point is colored naturally.

Note that the natural color for a new point p need not be unique. However, all $\prec$-maximal points in $\mathsf{validTops}(p)$ have the same up- and downsets and as such are indistinguishable from the game-theoretic point of view. Hence, a natural algorithm is free to choose any coloring from the set of equivalent candidates. Due to this observation we may abuse notation and frequently speak of 'the natural coloring' although there may be several.

The importance of natural colorings is given by the following theorem.

Theorem 3.3. *Natural algorithms are optimal strategies for on-line chain partitioning of up-growing semi-orders.*

Given an algorithm A we let $A(P)$ denote the set of chains generated, i.e., colors used, by A for the on-line order $\mathcal{P} = (P, \leqslant)$. The theorem follows from the next claim.

Claim 3.4. For any on-line algorithm A there is a natural on-line algorithm B such that $|B(P)| \leqslant |A(P)|$ for any up-growing semi-order $\mathcal{P}$.

Proof. Inductively we construct a sequence $A^{(k)}$ of algorithms such that $A^{(0)} = A$ and $A^{(k)}$ naturally colors the first k points of the up-growing semi-order $\mathcal{P}$. Moreover, $A^{(k+1)}$ is a modification of $A^{(k)}$ behaving exactly the same on first k points. Throughout the condition $|A^{(k)}(P)| \leqslant |A(P)|$ will be true.

Suppose $A^{(k)}$ is defined. We are going to define $A^{(k+1)}$. Let p be the $k+1$st point presented by Spoiler. There are three possibilities:

Case 1. $A^{(k)}$ colors p naturally. Define $A^{(k+1)}(p) = A^{(k)}(p)$ and $A^{(k+1)}(u) = A^{(k)}(u)$ for any future point u presented by Spoiler.

Case 2. $A^{(k)}$ assigns to p a new color γ, denoted $A^{(k)}(p) = \gamma$, although some natural algorithm would have used an old color $A^{(k)}(q)$. Define $A^{(k+1)}(p) = A^{(k)}(q)$ and extend the coloring $A^{(k+1)}$ by interchanging colors γ and $A^{(k)}(q)$ for points arriving later, i.e.,

$$A^{(k+1)}(u) = \begin{cases} A^{(k)}(q), & \text{if } A^{(k)}(u) = \gamma, \\ \gamma, & \text{if } A^{(k)}(u) = A^{(k)}(q), \\ A^{(k)}(u), & \text{otherwise,} \end{cases}$$

for any point u introduced by Spoiler after p. Clearly, $|A^{(k+1)}(P)| \leqslant |A^{(k)}(P)|$. It remains to show that $A^{(k+1)}$ defines a proper coloring of $\mathcal{P}$, i.e., that the sets $\{u \in P : A^{(k+1)}(u) = \gamma\}$ and $\{u \in P : A^{(k+1)}(u) = A^{(k)}(q)\}$ are chains in $\mathcal{P}$. The former is obvious, the latter follows from the fact that $u > p > q$ whenever $A^{(k+1)}(u) = A^{(k)}(q)$.

Case 3. $A^{(k)}$ assigns to p an old, non-natural color, i.e., there exist $r, q \in \mathsf{validTops}(p)$ such that q is $\prec$-maximal in $\mathsf{validTops}(p)$, $r \prec q$ in already presented poset and $A^{(k)}(p) = A^{(k)}(r)$. As in the previous case define $A^{(k+1)}(p) = A^{(k)}(q)$ and exchange colors $A^{(k)}(r)$ and $A^{(k)}(q)$ for all points appearing after p (see Figure 6). Clearly, p is naturally colored by $A^{(k+1)}$ and $|A^{(k+1)}(P)| = |A^{(k)}(P)|$. It remains to show that $A^{(k+1)}$ defines a proper coloring of $\mathcal{P}$. Obviously, the set $\{u \in P : A^{(k+1)}(u) = A^{(k)}(q)\}$ is a chain. Since $\mathcal{P}$ is a semi-order and $r \prec q$ at the moment when p is presented we get that $r \prec q$ in $\mathcal{P}$ at any future stage of the game and hence $r < u$ whenever $q < u$. This shows that the set $\{u \in P : A^{(k+1)}(u) = A^{(k)}(r)\}$ is a chain. $\square$

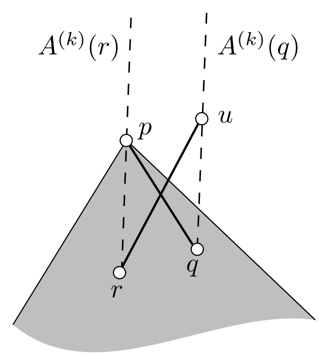


FIGURE 6. Exchanging colors $A^{(k)}(r)$ and $A^{(k)}(q)$

3.4. **The lower bound.** Let φ denote the golden ratio, i.e., $\varphi = \frac{1+\sqrt{5}}{2}$. In the next subsections we develop a strategy for Spoiler that forces Algorithm to use $\lfloor \varphi \cdot w \rfloor$ chains for an up-growing semi-order of width w. We assume that Algorithm responds forming a natural coloring. The strategy for Spoiler is presented in two steps. First (Subsection 3.5) we assume that the width of the semi-order $\mathcal{P}$ is given by a Fibonacci number F_{2k+1}. In this case Spoiler will enforce F_{2k+2} chains.

i	0	1	2	3	4	5	6	7	8	9	...
F_i	0	1	1	2	3	5	8	13	21	34	...

Since $F_{2k+2} = \lfloor \varphi \cdot F_{2k+1} \rfloor$ this will show the lower bound for orders of width F_{2k+1}. In a second step (Subsection 3.6) the strategy is analyzed for orders of arbitrary width.

3.5. **The lower bound: Fibonacci width.** Fix $k \geqslant 0$. Spoiler builds an order $\mathcal{P} = (P, \leqslant)$ of width F_{2k+1}. The height of $\mathcal{P}$ is at most 3, (for $F_1 = 1$ it is 1, for $F_3 = 2$ it is 2, for all others it is 3) therefore there is a canonical partitioning $P = A \cup B \cup C$ such that $A = \mathsf{Min}(P)$ and $B = \mathsf{Min}(P \setminus A)$. The points of $\mathcal{P}$ are presented in packages such that the presentation sequence has the following structure $P = (A, B_0, B_1, C_1, \ldots, B_{k-1}, C_{k-1}, B_k)$. Here, as suggested by notation $B_i \subseteq B$ and $C_i \subseteq C$ for all i. We next describe the $k+1$ phases of the construction of $\mathcal{P}$.

Phase 0. Spoiler presents the antichain A of size F_{2k+1}. Algorithm uses F_{2k+1} colors.

Phase 1. Spoiler presents a set B_0 of points, such that $|B_0| = F_{2k-1}$ and $B_0 > A$, this means that $b > a$ for all $b \in B_0$ and $a \in A$. Algorithm responds in a natural way and uses colors which have already been used in A. Let $A_0 \subseteq A$ be the set

of points whose colors have been used in B_0. Clearly, $|A_0| = F_{2k-1}$. Now Spoiler presents $B_1 > A_0$, $|B_1| = F_{2k-1}$. Algorithm has to introduce F_{2k-1} new colors for B_1 (see Figure 7).

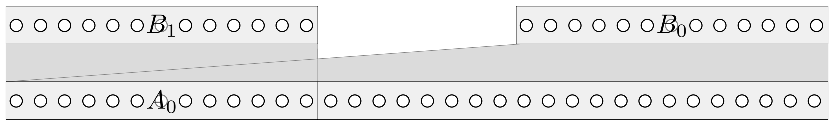


FIGURE 7. Order of width 34, phases 0–1, the sets B_0 and B_1 are of size 13 each

Phase 2. Spoiler presents a set C_1^0 of points such that $C_1^0 > A \cup B_1$ and $|C_1^0| = F_{2k-3}$. To color C_1^0 the natural Algorithm will use colors from a set $B_1^0 \subseteq B_1$, $|B_1^0| = F_{2k-3}$. Now Spoiler presents $C_1^1 > A \cup B_1^0$ of size $|C_1^1| = |C_1^0|$. Algorithm uses colors from a subset $A_1 \subseteq A \setminus A_0$, $|A_1| = |C_1^0|$. Spoiler presents $B_2 > (A_0 \cup A_1)$ with $|B_2| = |C_1^0|$. Algorithm has to introduce F_{2k-3} new colors for B_2 (see Figure 8).

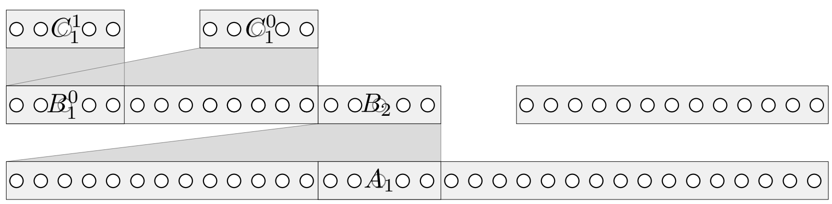


FIGURE 8. Order of width 34, phase 2, $|C_1^0| = |C_1^1| = |B_2| = 5$

First, let us informally describe the general $j+1$st phase which is similar to the second one. The phase starts with the set C_j^0 dominating $A \cup B_1 \cup \ldots \cup B_j$, which (since algorithm is natural) is *pulling* some colors from B_j. This starts a *cascade*: The set $B_j^0 \subseteq B_j$ whose colors have been used for C_j^0 makes room for a set C_j^1 which fetches colors from points of $B_j^1 \subseteq B_{j-1}$. This set B_j^1 makes room for C_j^2 which fetches colors from points of B_{j-2} .... This continues until C_j^j, this set is fetching colors from A which makes room for a set B_{j+1}. This last set of points has to receive new colors. All the sets introduced during this phase have size $F_{2k-2j-1}$. Figure 9 illustrates the 3rd phase.

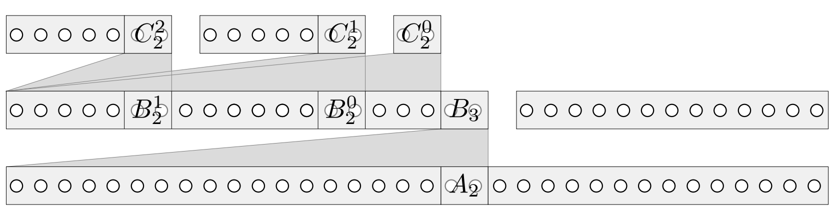


FIGURE 9. Order of width 34, phase 3, a cascade of sets of size 2

The following identity for Fibonacci numbers is crucial

$$F_1 + F_3 + F_5 + \ldots F_{2r+1} = F_{2r+2}. \tag{4}$$

This implies that all the sizes of all the sets B_i together do not exceed the width F_{2k+1}. Indeed $|B_0| + |B_1| + |B_2| + |B_3| + \ldots + |B_k| = F_{2k-1} + F_{2k-1} + F_{2k-3} +$

$F_{2k-5} + \ldots + F_1 = F_{2k-1} + F_{2k} = F_{2k+1}$. Similarly it follows that the sizes of all the sets C_j^{j-i} which are played over B_i sum up to the size of B_i.

Now comes a more formal and more detailed description of the $j+1$st phase.

Phase $j+1$. All sets defined in this phase have size $F_{2k-2j-1}$. Spoiler presents $C_j^0 > A \cup (B_1 \cup \ldots \cup B_j)$. To color C_j^0 the natural Algorithm uses colors from $B_j^0 \subseteq B_j$. Now for $i = 1, \ldots, j$ Spoiler plays C_j^i such that $C_j^i > A \cup (B_1 \cup \ldots \cup B_{j-i}) \cup (B_j^{i-1} \cup B_{j-1}^{i-2} \cup \ldots \cup B_{j-i+1}^0)$. Algorithm, being natural, uses colors from $B_j^i \subseteq B_{j-i}$ to color C_j^i. For C_j^j Algorithm uses colors from $A_j \subseteq A$. Now Spoiler presents $B_{j+1} > (A_0 \cup \ldots \cup A_j)$. This forces the Algorithm to introduce new colors since all the colors of predecessors of points in B_{j+1} have already been used for other points in B.

Since the set of downsets of the points from $\mathcal{P}$ is linearly ordered with respect to $\subseteq$ it is clear that $\mathcal{P}$ is an up-growing interval order. To see that $\mathcal{P}$ is a semi-order suppose that $\mathcal{P}$ is not $(\mathbf{3}+\mathbf{1})$-free, i.e., suppose there exist $a, b, c, d \in P$ such that $a < b < c$ and $d \parallel \{a, b, c\}$. Since $\mathcal{P}$ has height at most 3 and $A < C$, the only option is that $a \in A$, $b \in B$, $c \in C$ and $d \in B$. But for $b \in B_i$ and $d \in B_j$ it is easy to see that $a < d$ for $i \leqslant j$ and $d < c$ for $j < i$. Thus, $\mathcal{P}$ is $(\mathbf{3}+\mathbf{1})$-free so it is a semi-order.

The number of colors used by the natural algorithm is easily computed to be

$$|A| + |B_1| + \ldots + |B_k| = F_{2k+1} + F_{2k-1} + \ldots + F_1 = F_{2k+2}.$$

As mentioned in the beginning of the section $F_{2k+2} = \lfloor \varphi \cdot F_{2k+1} \rfloor$. This shows the lower bound for Fibonacci width.

3.6. **The lower bound: General width.** We now refine the technique to prove the lower bound for orders of arbitrary width. The idea for the strategy remains the same as before, all we have to do is to adjust the sizes for the sets played in the different phases.

For the analysis we introduce the following variables:

$$\begin{aligned} s_j &= |B_j| \quad \text{for } j > 0. \\ & \quad \text{The size of the sets in the cascade of phase } j. \\ r_j &= \lfloor \varphi \cdot w \rfloor - (s_0 + s_1 + \ldots + s_j) \\ & \quad \text{The missing number of colors after phase } j. \end{aligned}$$

We let $s_0 = w$ and define the sizes s_j recursively:

$$s_{j+1} = \min\{r_j, \lceil s_j\,(2-\varphi) \rceil\}.$$

Note that there is a k with $s_k > 0$ and $s_j = 0$ for $j > k$. Moreover, $s_{j+1} = \lceil s_j \cdot (2-\varphi) \rceil$ for $0 \leqslant j \leqslant k-2$, i.e., the value r_j in the min-function comes into the play only in the size of the last cascade. With the following claim we show that these sizes allow the sets of a cascade to fit where they belong. The reader may find it helpful to understand the statement of the Claim 3.5 as follows:

$$|A| \geqslant |B_0| + \sum_{j=1}^{k} |B_j|, \quad |B_i| \geqslant |B_{i+1}| + \sum_{j=i+1}^{k} |B_j|, \; i \geqslant 1.$$

Hence, the cascade can indeed be played without violating the width.

Claim 3.5. $s_i \geqslant s_{i+1} + r_i, \; i = 0, \ldots, k$.

Proof. From $\varphi^2 = \varphi + 1$ it follows that for every positive integer α:

$$\lfloor \varphi \cdot \alpha \rfloor = \alpha + \lfloor \alpha/\varphi \rfloor, \tag{5}$$

$$\alpha - \lfloor \alpha/\varphi \rfloor = \lceil \alpha(2-\varphi) \rceil. \tag{6}$$

The proof of the claim is by induction on i. For $i = 0$ we have

$$s_1 + r_0 = \lceil w(2-\varphi) \rceil + (\lfloor \varphi \cdot w \rfloor - w) = \lceil w(2-\varphi) \rceil + \lfloor w/\varphi \rfloor = w = s_0.$$

The induction step ($i = j+1$) is proved by following

$$\begin{aligned}
s_{j+1} - r_{j+1} &= s_{j+1} - (r_j - s_{j+1}) = 2s_{j+1} - r_j \overset{(ind)}{\geqslant} 2s_{j+1} - (s_j - s_{j+1}) \\
&= (s_{j+1} - \lfloor s_{j+1}/\varphi \rfloor) + (\lfloor s_{j+1}/\varphi \rfloor + s_{j+1}) + s_{j+1} - s_j \\
&\overset{(5),(6)}{=} \lceil s_{j+1}(2-\varphi) \rceil + \lfloor s_{j+1} \cdot \varphi \rfloor + s_{j+1} - s_j \\
&\geqslant s_{j+2} + \lfloor s_{j+1}(1+\varphi) \rfloor - s_j \\
&\overset{(*)}{=} s_{j+2} + \lfloor \lceil s_j(2-\varphi) \rceil (1+\varphi) \rfloor - s_j \\
&\geqslant s_{j+2} + \lfloor s_j(2-\varphi)(1+\varphi) \rfloor - s_j \\
&= s_{j+2}.
\end{aligned}$$

The inequality with (ind) uses induction hypothesis, in (*) we use the fact that $s_{j+1} = \lceil s_j(2-\varphi) \rceil$ (since $j+1$ is not the last step of the construction). The last equation follows from the fact that $(2-\varphi)(1+\varphi) = 1$. □

After phase k we have $\min\{r_k, \lceil s_k\,(2-\varphi) \rceil\} = 0$ which is only possible if $r_k = 0$. By definition this implies $w + s_1 + \ldots + s_k = \lfloor \varphi \cdot w \rfloor$ and since the construction is tailored as to force s_j new colors in phase j we conclude that Spoiler has in total forced $\lfloor \varphi \cdot w \rfloor$ colors.

Theorem 3.6. *The value of the chain partitioning game on the class up-growing semi-orders of width w is at least $\lfloor \varphi \cdot w \rfloor$, i.e.,*

$$\mathsf{val}(w) \geqslant \lfloor \varphi \cdot w \rfloor, \text{ where } \varphi = \frac{1+\sqrt{5}}{2}.$$

□

3.7. **The upper bound.** For a given $w > 1$ let $\mathcal{P} = (P, \leqslant)$ be an up-growing semi-order of width w which makes the natural algorithm use the maximal number of colors possible. Since natural algorithms are optimal $\mathsf{val}(w)$ colors are forced. In the following we will modify the order $\mathcal{P}$, the presentation sequence of $\mathcal{P}$ and the coloring Γ of $\mathcal{P}$ while sticking to the invariants:

- $\mathcal{P}$ is an up-growing semi-order of width w.
- Γ is a natural coloring of $\mathcal{P}$ forcing $\mathsf{val}(w)$ colors.

After having modified $\mathcal{P}$ and Γ as to obey a certain set of properties (Properties (**a**)–(**m**)) we can describe some of the conditions as a system of inequalities whose solution will show that $\mathcal{P}$ could not force more than $\lfloor \varphi \cdot w \rfloor$ colors. This shows that the lower bound from the previous section is also the upper bound for $\mathsf{val}(w)$.

Throughout the proof whenever we modify $\mathcal{P}$ or/and Γ to impose a property we want that $\mathcal{P}$ and Γ retains the properties imposed by previous modifications. We will not make this explicit in most of the cases but we will mention those cases where a previously imposed property fails to be valid after a new modification.

The first modification is easy. Intuitively, when considering $\mathcal{P}$ we may ignore points following the last point which got a new color.

Property (a).

$$P = (p_1, p_2, \ldots, p_n) \textit{ and } p_n \textit{ is } \Gamma\textit{-colored by a new color.}$$

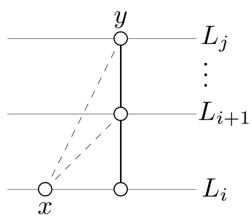
The next modification is more substantial. We are going to prove that the height of $\mathcal{P}$ can be restricted to be at most 3.

The canonical antichain partition $L_0, L_1, \ldots$ of $\mathcal{P} = (P, \leqslant)$ is defined by:

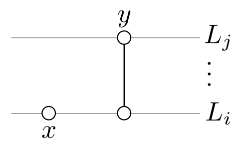
$$\begin{aligned} L_0 &= \mathsf{Min}(P), \\ L_{i+1} &= \mathsf{Min}(P \setminus (L_0 \cup \ldots \cup L_i)). \end{aligned}$$

Observation 3.7. *For a semi-order $\mathcal{P}$ the following two properties hold:*

(i) $x < y$ *whenever* $x \in L_i,\ y \in L_j,\ i \leqslant j-2$,
(ii) $x \prec y$ *whenever* $x \in L_i,\ y \in L_j,\ i < j$.



(i) forbidden $(\mathbf{3}+\mathbf{1})$



(ii) $x{\downarrow} \subsetneq y{\downarrow} \ \Rightarrow\ x \prec y$

FIGURE 10. Proof of Observation 3.7

Observation 3.8. *Let $\mathcal{P} = (P, \leqslant)$ be an up-growing semi-order. If Γ is a natural coloring of $\mathcal{P}$, then restricting Γ to $P \setminus \mathsf{Min}(P)$ yields a natural coloring of this smaller order.*

Proof. Pick $p \in P \setminus \mathsf{Min}(P)$. The proof is split into two cases:
1) $\Gamma(p) = \gamma,\ \mathsf{top}(\gamma) \in L_0$ in $\mathcal{P}$ before p is colored.
Since Γ is natural in $\mathcal{P}$ and $l_0 \prec l$ for any $l_0 \in L_0, l \in L_j$ $(j > 0)$ we deduce that $\mathsf{validTops}(p) \subseteq L_0$ in $\mathcal{P}$. In the smaller order $P \setminus L_0$ $\mathsf{validTops}(p) = \emptyset$ and p receives a new (natural) color γ.
2) $\Gamma(p) = \gamma,\ \mathsf{top}(\gamma) \in L_j$ in $\mathcal{P}$ before p is colored, $j > 0$.
Easily, $\mathsf{top}(\gamma)$ remains $\prec$-maximal in $\mathsf{validTops}(p)$ in the smaller order. □

Since the number of colors used by natural coloring Γ could decrease during the restriction provided by Obs. 3.8 we need some extra argument to reduce the height of $\mathcal{P}$ while retaining the number of colors used.

Property (b).

$$\mathsf{height}\,(\mathcal{P}) \leqslant 3.$$

Proof. Let h be the height of the point p_n, i.e., $p_n \in L_h$. Since $\mathcal{P}$ is an up-growing order $p_n{\uparrow} = \emptyset$. This, together with the fact, that $L_i < L_j$ whenever $i \leqslant j-2$ (Obs. 3.7.(i)), shows that L_j is empty for any $j \geqslant h+2$. We define a new poset $\mathcal{Q} = (Q, \leqslant_{\mathcal{Q}})$ which is a restriction of $\mathcal{P}$ to the three antichains $(L_{h-1} \cup L_h \cup L_{h+1})$.

Iterating Observation 3.8 we get that the restriction of Γ on $\mathcal{Q}$ yields a natural coloring. All we have to prove is that $|\Gamma(Q)| = |\Gamma(P)|$. Consider the point p_n. If the coloring Γ of $\mathcal{P}$ would have a chain which is completely contained in $P \setminus Q$, then by Observation 3.7(i) the color of this chain would have been valid for p_n. Since, according to (**a**), point p_n received a new color we conclude that the maximal element of every Γ chain is contained in $\mathcal{Q}$.

It follows that the height $\leqslant 3$ order $\mathcal{Q}$ forces as many colors as $\mathcal{P}$ and we may replace $\mathcal{P}$ by $\mathcal{Q}$. □

In the rest of the proof we may assume that $\mathcal{P}$ consists of at most three levels L_0, L_1 and L_2. From now on, we denote these levels by A, B and C, respectively.

Property (c).

$$\Gamma(A) \nsubseteq \Gamma(B \cup C),$$

i.e., some colors are used only on level A.

Proof. Suppose to the contrary that $\Gamma(A) \subseteq \Gamma(B \cup C)$. We claim that the level A is redundant. Indeed, by Observation 3.8 we may exclude A from P and retain the natural coloring Γ on $B \cup C$. Clearly, the number of colors used does not change. The proof that Γ remains a natural coloring on $B \cup C$ is as in Observation 3.7. □

It is easy to see that on the order of width w and height at most 2 Spoiler may force at most $\lfloor \frac{3}{2} w \rfloor$ colors. We have already seen that $\mathsf{val}(w) \geqslant \lfloor \frac{1+\sqrt{5}}{2} w \rfloor$, therefore for $w \geqslant 5$ the forcing poset $\mathcal{P}$ must have height 3.

Corollary 3.9.

(1) $A < C$,

(2) *New Γ-colors appear only on levels A and B, in other words, every chain has its minimal element in $A \cup B$.*

Proof. The fact that $A < C$ follows from Observation 3.7(i). To prove (2) assume that there exists $p \in C$ which has obtained a new color by Γ. Since $A < p$ and Γ is natural we deduce that $\Gamma(A) \subseteq \Gamma(B \cup C)$. This contradicts (**c**). □

Property (d).

$$|A| = w = \mathsf{width}\,(\mathcal{P})\,.$$

Proof. If $|A| < w$ then consider an additional set of points $A' = \{q_1, \ldots, q_k\}$ such that $|A'| + |A| = w$. We construct an extended poset $\mathcal{Q} = (Q, \leqslant)$, with $Q = (A', P)$ where points in A' are made incomparable to A, dominated by $B \cup C$ and played by Spoiler at the beginning, before the whole P (see Figure 11). Let $n_1, \ldots, n_k$ denote

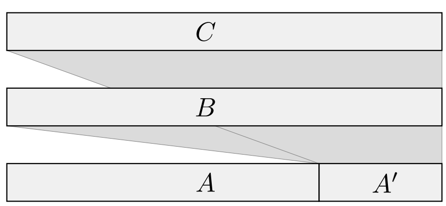


FIGURE 11. Property (**d**), poset $\mathcal{Q}$

the first k new Γ-colors introduced on level B or if such k colors do not exist let remaining n_i's be completely new Γ-colors. Define Δ, a coloring of $\mathcal{Q}$:

$$\Delta(p) = \begin{cases} \Gamma(p), & \text{if } p \in P, \\ n_i, & \text{if } p = q_i. \end{cases} \tag{7}$$

We argue that Δ is a natural coloring of $\mathcal{Q}$. Since $a' \preceq p$ for $a' \in A',\ p \in P$, colors from A' will be used only when there is no other option, exactly as Δ is defined. □

Property (e). *Points from A are played at the beginning of the construction.*

Proof. Let $P = (p_1, \ldots, p_i, a, p_{i+1}, \ldots, p_n)$ be the presentation sequence and $a \in A$. Consider the presentation sequence $P' = (a, p_1, \ldots, p_i,\ p_{i+1}, \ldots, p_n)$. Since $\mathcal{P}$ is presented in an up-growing way a moment of thought reveals that the natural coloring Γ of the old sequence is also natural for the new sequence. Iterating this yields a sequence where A is played at the beginning. $\square$

Corollary 3.10. *$p_n \in B$, $p_n{\downarrow} \subsetneq A$.*

Proof. By (**a**) and Corollary 3.9.(2) $p_n \in A \cup B$. But if $p_n \in A$ by (**e**) whole $\mathcal{P}$ would be an antichain, contradiction for $w > 1$ with $|\Gamma(P)| = \mathsf{val}(w) \geqslant \lfloor \frac{1+\sqrt{5}}{2} w \rfloor$. If in turn $p_n{\downarrow} = A$ then by (**a**) there would be no Γ-top in A contradicting (**c**). $\square$

The next two modifications deal with the level B. First, we define the set

$$B_0 = \{p \in B :\ \Gamma(p) \in \Gamma(A)\},$$

i.e., B_0 is the set of points of B which receive an old color.

Property (f). *B_0 is played right after A, moreover, every point in B_0 is above all of A, i.e., $A < B_0$.*

Proof. Let $P = (A, p_1, \ldots, p_i, b, p_{i+1}, \ldots, p_n)$ be the presentation sequence and let b be the last element of this sequence which is in B_0.

Case 1. Suppose that $b{\uparrow} = \emptyset$. Let $\mathcal{Q} = (Q, \leqslant_{\mathcal{Q}})$ be as $\mathcal{P}$ but with the modified sequence $Q = (A, b, p_1, \ldots, p_i, p_{i+1}, \ldots, p_n)$ and if necessary also with an augmented order relation so that b is above all of A.

Obviously $\mathcal{Q}$ is an up-growing order. It is evident that the set of downsets $\{q{\downarrow}_{\mathcal{Q}} : q \in Q\}$ is linearly ordered by inclusion, therefore $\mathcal{Q}$ is an interval order. Suppose that $\mathcal{Q}$ is not a semi-order, then, point b must contribute to a $(\mathbf{3}+\mathbf{1})$. Since b has gained additional comparabilities it can't be the singleton, but $b \in B$ and has no successor, hence it can't be in the 3-chain either. The coloring Γ is also natural for $\mathcal{Q}$.

Case 2. Now assume that $b{\uparrow} \neq \emptyset$. The above construction for $\mathcal{Q}$ may lead to a $(\mathbf{3}+\mathbf{1})$. Therefore we have to do something different.

Note that $b{\downarrow} \subseteq p_n{\downarrow}$ (see Figure 12), otherwise there would be a $(\mathbf{3}+\mathbf{1})$ configuration.

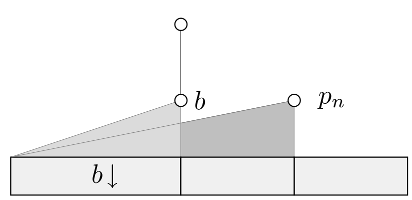


FIGURE 12. Property (**f**), $b{\downarrow} \subseteq p_n{\downarrow}$

We now consider the presentation sequence $(A, p_n, p_1, \ldots, p_i, b, p_{i+1}, \ldots, p_{n-1})$ with p_n immediately after A and augment the order relation so that p_n is above all of A. The same reasoning as in the previous case shows that this order $\mathcal{Q}$ is an up-growing semi-order. The coloring Γ, however, fails to be natural for $\mathcal{Q}$. Define a new coloring Δ, interchanging colors $\Gamma(b)$ and $\Gamma(p_n)$ on levels B and C:

$$\Delta(p) = \begin{cases} \Gamma(b), & \text{if } p = p_n, \\ \Gamma(p_n), & \text{if } \Gamma(p) = \Gamma(b) \text{ and } p \notin A, \\ \Gamma(p), & \text{otherwise.} \end{cases}$$

Claim 3.11. Δ is a natural coloring for $\mathcal{Q}$.

The claim is easily established if we show that the new color which is used for point b is justified by $\mathsf{validTops}_{\Delta,\mathcal{Q}}(b) = \emptyset$.

Recall that we have chosen b as the last point of B_0 in the sequence of $\mathcal{P}$, i.e., the last point from B which received an old color by Γ. If b had only one available old color, i.e., if $|\mathsf{validTops}_{\Gamma,\mathcal{P}}(b)| = 1$, then we are done since this color has been taken by p_n in the Δ coloring of $\mathcal{Q}$.

Now suppose that $|\mathsf{validTops}_{\Gamma,\mathcal{P}}(b)| > 1$, this will be lead to a contradiction with the naturality of coloring Γ on $\mathcal{P}$. Our assumption implies that after coloring b there was a $a \in b{\downarrow_{\mathcal{P}}}$ whose Γ-color was not used in B. When coloring p_n this color was not available since p_n got a new color and $b{\downarrow_{\mathcal{P}}} \subseteq p_n{\downarrow_{\mathcal{P}}}$. Hence there is a $c \in C$ with $\Gamma(c) = \Gamma(a)$. Now consider $d \in A$ such that $\Gamma(d)$ was not used for a point in $B \cup C$, point c exists by (**c**). Note that $d \notin p_n{\downarrow}$ but $a \in p_n{\downarrow}$, hence $a \prec d$. Also $d \leqslant c$ because $d \in A$ and $c \in C$ (Observation 3.7). Together this shows that a natural coloring would prefer $\Gamma(d)$ to $\Gamma(a)$ when coloring c. This contradiction completes the proof of the claim.

Repeating this process we eventually reach the presentation sequence $(A, B_0, \ldots)$. Note that the final $\mathcal{P}$ of this modification may fail to give a new color to the latest point, this can be repaired by going through modification (**a**) again. $\square$

Since $p_n{\downarrow} \subsetneq A$ and $\mathcal{P}$ is a semi-order, i.e., $(\mathbf{3}+\mathbf{1})$-free, we can deduce that $B_0{\uparrow} = \emptyset$, (see Figure 13). This implies that after B_0 there is a group of points B_1 from the level B. We continue and split the sequence P as follows:

$$P = (A, B_0, B_1, C_1, \ldots, B_k, C_k, B_{k+1}).$$

Sets B_i, C_i are non-empty groups of points from levels B and C, respectively. The last block is a B-block because $p_n \in B$. The value $k \geqslant 0$ depends on $\mathcal{P}$ and its presentation. The following observation allows to get additional structure on the B-blocks.

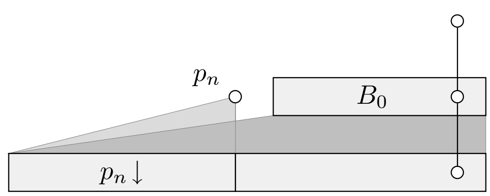


FIGURE 13. $B_0{\uparrow} = \emptyset$ because of forbidden $(\mathbf{3}+\mathbf{1})$

Observation 3.12 (Transposition of points)**.** *Let $P = (p_1, \ldots, p_n)$ be the presentation of an up-growing semi-order of height at most* 3 *and let Γ be a corresponding natural coloring. If*

(1) *both p_i and p_{i+1} got a new Γ-color, or*

(2) *$p_i \in C$, $p_{i+1} \in B$, and p_i did not get its color from an element of $p_{i+1}{\downarrow}$,*

then Γ remains a natural coloring for the up-growing presentation

$$P' = (p_1, \ldots, p_{i-1}, p_{i+1}, p_i, p_{i+2} \ldots, p_n).$$

Proof. In both cases p_i and p_{i+1} are incomparable, hence, the presentation P' of $\mathcal{P}$ is a linear extension, i.e., up-growing. In the situation (1) we have $\mathsf{validTops}(p_i) = \emptyset$ and $\mathsf{validTops}(p_{i+1}) = \emptyset$ which is invariant under transposition. For (2) we note that

the $\prec$-maximal elements of $\mathsf{validTops}(p_i)$ and $\mathsf{validTops}(p_{i+1})$ remain unaffected by the transposition. $\square$

Property (g).

$$b{\downarrow} \subsetneq b'{\downarrow} \text{ for any } b \in B_i,\ b' \in B_j,\ 1 \leqslant i < j \leqslant k+1.$$

Proof. Let $b \in B_i,\ b' \in B_j$ be a pair which does not satisfy the property. Since $\mathcal{P}$ is an interval order $b'{\downarrow} \subseteq b{\downarrow}$. We show how to move b' from B_j to B_i using a sequence of transpositions. Iterating this procedure we reach a situation as strived for.

Let p be the point that precedes b' in $\mathcal{P}$, i.e., $P = (\ldots, b, \ldots, p, b', \ldots)$. We show how to swap p and b'. There are two possibilities: $p \in B$ or $p \in C$. Assume first that $p \in B$. Recall from (**f**) that points b, p and b' received a new Γ-color. According to Observation 3.12 we may swap p and b'.

Now assume that $p \in C$. Let q be the point from which p got its color. If $q \in B$ then it is incomparable with b' and we are allowed to swap p and b' by Observation 3.12. Now assume that $q \in A$, we claim that q and b' are incomparable. To verify this recall that b got a new color, hence $q \notin b{\downarrow}$. Since by assumption $b'{\downarrow} \subseteq b{\downarrow}$ we also have $q \notin b'{\downarrow}$. Hence, again we may swap p and b' by Observation 3.12. $\square$

Property (h).

$$b{\downarrow} = b'{\downarrow} \text{ for any } b, b' \in B_i,\ i = 1, \ldots, k+1.$$

Proof. Fix $i > 0$ and suppose there exist $b, b' \in B_i$ such that $b{\downarrow} \neq b'{\downarrow}$. Choose b' with a maximal set of predecessors among the points in B_i. Define order $\mathcal{Q} = (P, \leqslant_{\mathcal{Q}})$ with $\leqslant_{\mathcal{Q}}$ being the same as $\leqslant_{\mathcal{P}}$ except for the points from B_i, whose sets of predecessors in $\leqslant_{\mathcal{Q}}$ are equal to $b'{\downarrow}_{\mathcal{P}'}$. We claim that we may take $\mathcal{Q}$ as the new $\mathcal{P}$. We prove the two non-obvious conditions:

1) $\mathcal{Q}$ is $(\mathbf{3}+\mathbf{1})$-free. Suppose to the contrary there exist $q_1, q_2, q_3, q_4 \in P$ such that $q_1 < q_2 < q_3$ and q_4 is incomparable with the other three points in $\mathcal{Q}$ (see Figure 14). Since $\mathcal{P}$ is $(\mathbf{3}+\mathbf{1})$-free, the forbidden configuration must be formed by one of the extra edges in $\mathcal{Q}$ and obviously $q_2 \in B_i$. Recall that $A < C$ and therefore $q_4 \in B$. The point q_1 became a predecessor of q_2 in $\mathcal{Q}$ because $q_1 \in p{\downarrow}$ in $\mathcal{P}$ for some p which is in B_i together with q_2. Since property (**g**) was already established for $\mathcal{P}$ we can conclude that $q_4 \in B_1 \cup \ldots \cup B_{i-1}$ (otherwise we would have $q_1 < q_4$). Since $\mathcal{P}$ fulfills (**g**) we deduce that $q_4{\downarrow} \subsetneq q_2{\downarrow}$ in $\mathcal{P}$. Hence, in $\mathcal{P}$ there exists $a \in A$ smaller than q_2 but incomparable to q_4. It is now easy to check that the quadruple $\{a, q_2, q_3, q_4\}$ forms a $(\mathbf{3}+\mathbf{1})$ configuration in $\mathcal{P}$, a contradiction.

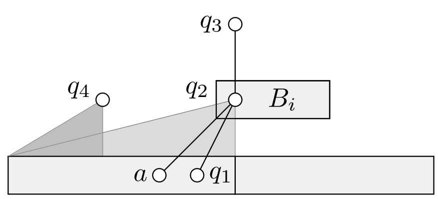


FIGURE 14. Property (**h**), $\mathcal{Q}$ is $(\mathbf{3}+\mathbf{1})$-free

2) Γ is natural on $\mathcal{Q}$. The modification did increase the set of predecessors of some points B_i. Consider a point $b \in B_i$ with $b{\downarrow}_{\mathcal{Q}} \supsetneq b{\downarrow}_{\mathcal{P}}$. By definition $b{\downarrow}_{\mathcal{Q}} = b'{\downarrow}_{\mathcal{P}}$ for some $b' \in B_i$. Since the elements of the block are presented consecutively and

$\mathsf{validTops}_{\mathcal{P}}(b') = \emptyset$ the increase of $b{\downarrow}$ did not catch new valid predecessors, i.e., $\mathsf{validTops}_{\mathcal{Q}}(b) = \emptyset$. This shows that in $\mathcal{Q}$ point b is colored naturally by Γ.

The modification of points in B_i changed the $\prec$-ordering in $\mathcal{Q}$. We need an argument showing that Γ remains natural on points from level C. But the order $\prec_{\mathcal{P}}$ is an extension of $\prec_{\mathcal{Q}}$, therefore every $\prec_{\mathcal{P}}$-maximal point in $\mathsf{validTops}(p)$ is also $\prec_{\mathcal{Q}}$-maximal. $\square$

Note that by now the sets A and B with its refinement $B_0, \ldots, B_{k+1}$ have the properties of the corresponding sets in the lower bound construction. With the next few modifications we establish some properties on the level C of the poset. Unfortunately, these modifications are more complicated and technical.

Recall that poset $\mathcal{P}$ is presented as $P = (A, B_0, B_1, C_1, \ldots, B_k, C_k, B_{k+1})$. Let $p \in C$. Define $j(p)$ as the largest number such that at least one point from the set $B_{j(p)}$ is covered by p, i.e.,

$$j(p) = \max\{i : B_i \cap p{\downarrow} \neq \emptyset\}.$$

Recall from Fig. 13 that $B_0{\uparrow} = \emptyset$, hence, B_0 is pairwise incomparable with C and $j(p) \geqslant 1$ for all $p \in C$. Figure 15 illustrates the fact that $(\mathbf{g})$ and the $(\mathbf{3}+\mathbf{1})$-freeness imply

$$B_1 \cup \ldots \cup B_{j(p)-1} \subseteq p{\downarrow} \text{ for } j(p) \geqslant 2. \tag{8}$$

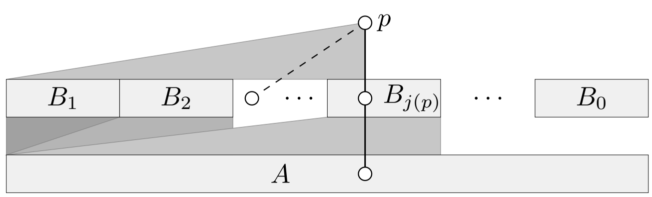


FIGURE 15. $B_1 \cup \ldots \cup B_{j(p)-1} \subseteq p{\downarrow}$ for $j(p) \geqslant 2$

Definition 3.13.

- A point $p \in C$ with $j = j(p)$ is a *pulling point on* B_j iff $\Gamma(p) \in \Gamma(B_j)$. The set of pulling points on B_j is denoted by W_j.
- A point $p \in C$ with $j = j(p)$ which is not pulling is a *cascading point on* B_j. The set of cascading points on B_j is denoted by K_j.
- Let U_j be the set of points from C with $j(p) = j$, i.e.,
$$U_j = K_j \cup W_j.$$
- Let F_j be the subset of points from B_j whose colors are used in C, i.e.,
$$F_j = \{b \in B_j : \Gamma(b) \in \Gamma(C)\}.$$
- Let F be the subset of points from A whose colors are used in $B \cup C$, i.e.,
$$F = \{a \in A : \Gamma(a) \in \Gamma(B \cup C)\}.$$
- Denote the sizes of the introduced sets by small letters, i.e.,
$$b_j = |B_j|, \quad w_j = |W_j|, \quad k_j = |K_j|, \quad f_j = |F_j|, \quad f = |F|.$$

Property (i).

$$B_i < W_i$$

and all points from W_i are played at the beginning of C_i, for $i = 1, \ldots, k$.

Proof. Let $p \in C_j$ $(j \geqslant i)$ be any pulling point on B_i. We construct $\mathcal{Q}$ such that p is the first point of C_i and p dominates all of B_i. By iterating this process (**i**) can be satisfied.

Let $\mathcal{Q}$ be obtained from $\mathcal{P}$ by adding comparabilities so that $B_i \subseteq p{\downarrow}$ and let the presentation be $Q = (A, B_0, \ldots, B_i, p, C_i, \ldots, B_j, C_j \setminus \{p\}, \ldots, B_{k+1})$. By definition of $i = j(p)$ all predecessors of p are in $A \cup B_1 \cup \ldots \cup B_i$ (in fact this set equals $p{\downarrow}_{\mathcal{Q}}$). Since all of them are already played at the time p arrives $\mathcal{Q}$ is an up-growing order. It is easy to see that this modification does not produce $(\mathbf{3}+\mathbf{1})$ configuration, thus $\mathcal{Q}$ remains a semi-order. Note that all points in B_i are made $\prec_{\mathcal{Q}}$-maximal in $\mathsf{validTops}(p)$. Hence, the coloring Γ remains natural for p in $\mathcal{Q}$. Modifying the set of predecessors of p can make some points in B_j $\prec$-incomparable in $\mathcal{Q}$ although they were comparable with respect to $\prec_{\mathcal{P}}$. But again the fact that $\prec_{\mathcal{Q}} \subseteq \prec_{\mathcal{P}}$ guarantees that Γ is natural for $\mathcal{Q}$. □

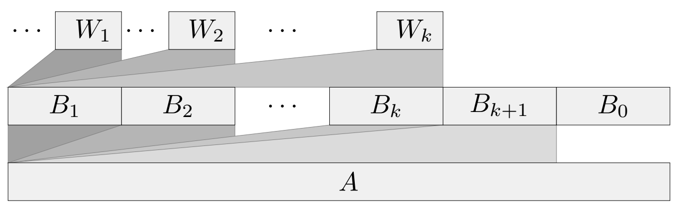


FIGURE 16. Placing of W_i's (pulling points) in $\mathcal{P}$

Property (j). *A cascading point $p \in K_i$ dominates precisely those points of B_i whose color have already been used on level C when p is presented, i.e., for $p \in K_i$ and $b \in B_i$ we have $b < p$ iff there is a $c \in C$ with $\Gamma(c) = \Gamma(b)$ and c precedes p in the presentation order.*

Proof. For a given p add all the necessary comparabilities at once. It is easy to see that $\mathcal{P}$ remains an up-growing semi-order. The naturality of Γ on $\mathcal{Q}$ again follows from the fact that $\prec_{\mathcal{Q}} \subseteq \prec_{\mathcal{P}}$. □

Corollary 3.14. *For any two cascading points $k, k' \in K_i$*

$$k{\downarrow} \subseteq k'{\downarrow} \quad \textit{whenever } k \textit{ precedes } k' \textit{ in } \mathcal{P}.$$

Property (k).

$$\mathsf{width}\,(B_i \cup U_i) = |B_i| \quad \textit{for } i = 1, \ldots, k.$$

Proof. Trivially $\mathsf{width}\,(B_i \cup U_i) \geqslant |B_i|$. Suppose that $\mathsf{width}\,(B_i \cup U_i) > B_i$. Since $B_i < W_i$ (by (**i**)) and $|B_i| \geqslant |W_i|$ (by the definition W_i takes colors from B_i) we have $U_i \neq W_i$ therefore $K_i \neq \emptyset$. Pick $p \in K_i$ having the minimal set of predecessors. Define $\mathcal{Q} = (Q, \leqslant)$ such that p keeps no predecessors in B_i, i.e., $p{\downarrow}_{\mathcal{Q}} = p{\downarrow}_{\mathcal{P}} \setminus B_i$. It is easy to see that $\mathcal{Q}$ remains a semi-order and that Γ is natural on $\mathcal{Q}$. Having deleted comparabilities, however, we may have increased the width. The following claim shows that this is not the case.

Claim 3.15. $\mathsf{width}\,(\mathcal{Q}) = \mathsf{width}\,(\mathcal{P})$.

Proof. Suppose for the contrary that $\mathsf{width}(\mathcal{Q}) > \mathsf{width}(\mathcal{P})$. Let X be a maximum antichain in $\mathcal{Q}$. Clearly, X contains p and some $r \in B_i$ with $r <_{\mathcal{P}} p$.

Since p dominates all points in A we deduce that $X \subseteq B \cup C$. Furthermore, since $r < (U_i \cup U_{i+1} \cup \ldots \cup U_k)$ and $p > (B_1 \cup \ldots \cup B_{i-1})$ in $\mathcal{Q}$ (see Figure 17) we deduce that

$$X \subseteq B_0 \cup (B_i \cup \ldots \cup B_{k+1}) \cup (U_1 \cup \ldots U_{i-1}) \text{ in } \mathcal{Q}. \tag{9}$$

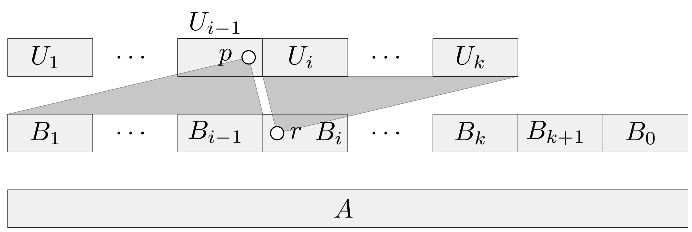


FIGURE 17. $r < (U_i \cup \ldots \cup U_k)$ and $p > (B_1 \cup \ldots \cup B_{i-1})$ in $\mathcal{Q}$

Consider $X' = X \setminus \{p\}$. This is an antichain in $\mathcal{P}$. We will show that $|X'| < \mathsf{width}(\mathcal{P})$.

From $X \cap (B_i \cup U_i) \subseteq X' \cap (B_i \cup U_i) \subseteq B_i$ and the assumption $\mathsf{width}(B_i \cup U_i) > |B_i|$ it follows that

$$\mathsf{width}(X' \cap (B_i \cup U_i)) < \mathsf{width}(B_i \cup U_i) \text{ in } \mathcal{P}. \tag{10}$$

From (9) it follows that

$$X' \setminus (B_i \cup U_i) \text{ is pointwise incomparable to } (B_i \cup U_i) \text{ in } \mathcal{P}. \tag{11}$$

From (10) and (11) we obtain

$$\begin{aligned} |X'| &= |X' \setminus (B_i \cup U_i)| + |X' \cap (B_i \cup U_i)| \\ &< |X' \setminus (B_i \cup U_i)| + \mathsf{width}(B_i \cup U_i) \leqslant \mathsf{width}(\mathcal{P}), \end{aligned}$$

hence $|X| \leqslant \mathsf{width}(\mathcal{P})$. □

If p became a pulling point on B_{i-1} in $\mathcal{Q}$ there can be a violation of Property (**i**) (p could be not as (**i**) states). In this case (**i**) needs to be reestablished. In the case $i = 1$ point p in $\mathcal{Q}$ falls to level B and since it dominates A and has a Γ-color from A property (**f**) needs to be reestablished.

The size of U_i has decreased in $\mathcal{Q}$. Repeating this process we eventually establish (**k**). □

Corollary 3.16.

$$k_i \leqslant f_i \quad \textit{for } i = 1, \ldots, k.$$

Proof. Since $K_i \cup (B_i \setminus K_i\!\downarrow)$ is an antichain in $B_i \cup U_i$ we deduce with (**k**) that

$$|K_i| \leqslant |K_i\!\downarrow \cap B_i|. \tag{12}$$

The statement now follows from the fact that $|K_i\!\downarrow \cap B_i| \leqslant f_i$ as any cascading point $p \in K_i$ may only cover points from B_i whose colors were already used in level C. □

The next Property is the workhorse of the calculations for the upper bound. Loosely speaking it ensures that cascading points from B_i use colors from points in B_{i-1} (for $i \geqslant 2$) which implies that the only points taking colors from A are those from $B_0 \cup K_1$. This fact will be used to bound the number of colors reused in $\mathcal{P}$.

Property (l).

$$\Gamma(B_i) \not\subseteq \Gamma(C) \quad \textit{for } i = 1, \ldots, k,$$

i.e., not all Γ*-colors from* B_i *are reused in* C*.*

Proof. Let $\mathcal{P} = (P, \leqslant)$ and its coloring Γ be a pair satisfying (**a**)–(**k**) and failing to satisfy (**l**). Let s be the smallest index for which (**l**) does not hold:

$$s = \min \{ i \geqslant 1 : \ \Gamma(B_i) \subseteq \Gamma(C) \} .$$

We define $\mathcal{Q}$ and its natural coloring Δ such that the pair $(\mathcal{Q}, \Delta)$ satisfies (**a**)–(**k**), the size of the set B_s in $\mathcal{Q}$ is strictly smaller than the size of B_s in $\mathcal{P}$ and (**l**) holds for 'earlier' B-blocks, i.e., $\Gamma(B_i) \not\subseteq \Gamma(C)$ for $i = 1, \ldots, s-1$. We repeat this process until all bad B-blocks are removed and (**l**) is satisfied.

Let $b_s \in B_s$ be the point whose color was used on C last and let $p \in C$ be the point colored with $\Gamma(b_s)$. Depending on properties of the point p we split the rest of the proof into three cases.

Case 1. p is a pulling point on B_s.

Since p is a pulling point which took the last available color from B_s we deduce from (**i**) and (**k**) that $|B_s| = |W_s|$ and $K_s = \emptyset$.

Define the poset $\mathcal{Q}$ in which points b_s and p are replaced by a point q_s:

$$Q = (A, B_0, \ldots, B_s \setminus \{b_s\}, C_s \setminus \{p\}, \ldots, B_{k+1}, q_s),$$

$$q{\downarrow_{\mathcal{Q}}} = \begin{cases} B_{k+1}{\downarrow_{\mathcal{P}}}, & \text{if } q = q_s, \\ q{\downarrow_{\mathcal{P}}} \setminus \{b_s\}, & \text{otherwise.} \end{cases}$$

$$\Delta(q) = \begin{cases} \Gamma(q), & \text{if } q \neq q_s, \\ \Gamma(b_s), & \text{if } q = q_s. \end{cases}$$

Clearly, Δ and Γ use the same number of colors as $\Delta(q_s) = \Gamma(b_s) = \Gamma(p)$ and $\mathcal{Q}$ remains a semi-order. Since $\Gamma(b_s)$ and $\Delta(q_s)$ are new colors for Γ and Δ the coloring Δ is natural for $\mathcal{Q}$. To prove that $\mathsf{width}\,(\mathcal{Q}) = \mathsf{width}\,(\mathcal{P})$ first observe that $\mathsf{width}\,(A \cup B)$ in $\mathcal{Q}$ is the same as $\mathsf{width}\,(A \cup B)$ in $\mathcal{P}$. Thus, it suffices to show that (**k**) holds in $\mathcal{Q}$. For $i \neq s$ the set $B_i \cup U_i$ is left unchanged and (**k**) follows from $\mathcal{P}$. Since $B_s < W_s = U_s$, exactly two antichains (B_s and W_s) realize the width of $B_s \cup W_s$ in $\mathcal{P}$. Deleting b_s and p decreases the size of both of them. This proves (**k**) for B_s.

Case 2. p is a cascading point on B_t $(t > s)$ and there are no cascading points on B_s.

Recall that $\Gamma(p) \in \Gamma(B_s)$ meaning that not all colors from $\Gamma(B_s)$ are used by W_s. Combining this with $K_s = \emptyset$ we have

$$|B_s| > |W_s| = |U_s|. \tag{13}$$

Define $\mathcal{Q}$ and Δ as in the previous case, i.e., points b_s and p are replaced by point q_s in B_{k+1}. Only the proof for (**k**) has to be adapted. For $i \notin \{s, t\}$ property (**k**) is obvious. Since we removed $p \in U_t$ while B_t was left unchanged (**k**) holds in B_t. The fact that (**k**) holds for $i = s$ follows directly from (13) and (**i**).

Case 3. p is a cascading point on B_t $(t > s)$ and there are some cascading points on B_s, i.e., $p \in K_t$ and $K_s \neq \emptyset$.

The previous construction may not be applied as it can happen that $|B_s| = |U_s|$ in $\mathcal{P}$ and reducing the size of B_s spoils (**k**) in $B_s \cup U_s$. Let k_s be the last cascading point from K_s and let b_{s-1} be a point from which k_s got its Γ-color. Note that $b_{s-1} \in B_{s-1}$ by the choice of s.

Case 3.1. p precedes k_s in $\mathcal{P}$ (in order of presentation).

According to Property (**j**) $b_s < k_s$. Define $\mathcal{Q}$ in which points b_s and k_s are replaced with q_s:

$$Q = (A, B_0, \dots, B_s \setminus \{b_s\}, C_s \setminus \{k_s\}, \dots, B_{k+1}, q_s),$$

$$q{\downarrow_{\mathcal{Q}}} = \begin{cases} B_{k+1}{\downarrow_{\mathcal{P}}}, & \text{if } q = q_s, \\ q{\downarrow_{\mathcal{P}}} \setminus \{b_s\}, & \text{otherwise.} \end{cases}$$

$$\Delta(q) = \begin{cases} \Gamma(q), & \text{if } q \notin \{q_s, p\}, \\ \Gamma(b_s), & \text{if } q = q_s. \\ \Gamma(b_{s-1}), & \text{if } q = p. \end{cases}$$

Clearly, Δ and Γ use the same number of colors and $\mathcal{Q}$ remains a semi-order. We are going to argue that Δ is natural on $\mathcal{Q}$. Note that any cascading point $k \in K_{s-1}$ dominating b_{s-1} in $\mathcal{P}$ was played after k_s was introduced (otherwise, a natural coloring Γ would assign color $\Gamma(b_{s-1})$ to k). To prove that coloring $\Delta(p) = \Gamma(b_{s-1})$ is natural observe that at the moment when p arrives:

- there is no point from $B_{s+1}, \dots, B_t$ in $\mathsf{validTops}(p)$ in $\mathcal{Q}$ as there were no such in $\mathsf{validTops}(p)$ in $\mathcal{P}$,
- all the colors from $\Gamma(B_s)$ have already been used in C (b_s does not exist in $\mathcal{Q}$),
- points from B_{s-1} whose colors have not been used in C have all equal sets of predecessors (**h**) and successors (namely W_{s-1} and all points from the U_i with $i \geqslant s$ which have already been presented).

The fact that Δ colors naturally all other points in $\mathcal{Q}$ is immediate from the property of Γ.

To prove that the width of $\mathcal{Q}$ does not exceed the width of $\mathcal{P}$ we argue that (**k**) holds for $B_s \cup U_s$ in $\mathcal{Q}$, i.e., $\mathsf{width}\,(B_s \cup U_s) = |B_s|$ in $\mathcal{Q}$. From (**j**) and definition of b_s it follows that every cascading point k on B_s dominating b_s dominates the whole B_s. Hence, any maximal antichain in $B_s \cup U_s$ contains exactly one of the deleted points b_s or k_s.

Case 3.2. p is played after k_s in $\mathcal{P}$.

As in the previous case, we head for the poset $\mathcal{Q}$ in which points b_s and k_s are replaced with q_s. This time, however, a more subtle construction is needed in order to retain the coloring from $\mathcal{P}$. Indeed, $\mathcal{P}$ may have a point $k \in K_{s-1}$ dominating b_{s-1} which is played before p. If k would be left in $\mathcal{Q}$ without further modifications, a natural color assigned to k would come from b_{s-1} and not from earlier B-blocks.

Define the sequence of points $b_s, k_s, b_{s-1}, k_{s-1}, \dots, k_r, b_{r-1}$ ($r \geqslant 1$) as follows. Points b_s and k_s are already defined. Let b_i be the point from which k_{i+1} got its Γ-color. Let k_i be the first point in order presentation from K_i dominating b_i (if such does not exist sequence ends on b_i and $i = r - 1$, see Figure 18). From the minimality of s (B_s being the first B-block violating (**l**)) we have $b_i \in B_i$ for $i \geqslant 1$. Observe also that if $k_1 \in K_1$ is defined then it takes Γ-color from A (by the definition of K_1 there is no valid top for k_1 in B_1 and by (**c**) there must be some in A). Thus, in this setting $b_0 \in A$, see Figure 19.

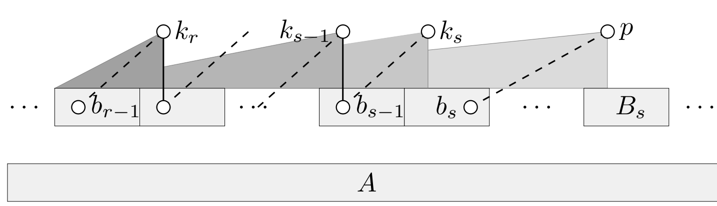


FIGURE 18. Property (**l**), Case 3.2.1, $r > 1$

Case 3.2.1. $r > 1$.

Observe that the cascade of points $k_s, \dots, k_r$ does not pull any color from A to C. Intuitively, such a sequence of moves is redundant and may be omitted. We define $\mathcal{Q}$ in which the cascade $k_s, \dots, k_r$ and the point b_s are not presented but in exchange a new point q_s is played. All remaining cascading points from K_i ($i = s-1, \dots, r$) are made incomparable with b_i, i.e.,

$$k{\downarrow_{\mathcal{Q}}} = k{\downarrow_{\mathcal{P}}} \setminus \{b_i\}, \text{ for } k \in K_i \setminus \{k_i\}, \ i = s-1, \dots, r.$$

Suppose that there was a $k \in K_i$, $k \neq k_i$ such that $k{\downarrow_{\mathcal{P}}} \cap B_i = \{b_i\}$, then by Corollary 3.14 $k_i{\downarrow_{\mathcal{P}}} \cap B_i = \{b_i\}$ as well. This, however, cotradicts the local width Property (**k**) of $B_i \cup U_i$ in $\mathcal{P}$. In conclusion we know that $k{\downarrow_{\mathcal{Q}}} \cap B_i \neq \emptyset$ for $k \in K_i \setminus \{k_i\}, \ i = s-1, \dots, r$, i.e., every $k \in K_i$ remains a cascading on B_i in $\mathcal{Q}$ as well.

We define a natural coloring Δ of $\mathcal{Q}$ as follows: $\Delta(q_s) = \Gamma(b_s)$, $\Delta(p) = \Gamma(b_{s-1})$, $\Delta(q) = \Gamma(q)$ for $q \notin \{b_s, b_{s-1}\}$. Note that Δ uses in $\mathcal{Q}$ the same number of colors as Γ in $\mathcal{P}$—the deleted points $k_s, \dots k_t$ used in $\mathcal{P}$ old Γ-colors from level B and the color of b_s is used on q_s.

We show that (**k**) holds for $B_r, \dots, B_{s-1}$ in $\mathcal{Q}$. Fix $i \in \{t, \dots, s-1\}$. Since $|B_i| = \mathsf{width}\,(B_i \cup U_i)$ in $\mathcal{P}$, there exists an injective function $f : U_i \to B_i$ such that $u > f(u)$ for any $u \in U_i$ in $\mathcal{P}$. If there is no $k \in U_i$ such that $f(k) = b_i$ or it exists and $k > b_i$ also in $\mathcal{Q}$ then f witnesses (**k**) for $B_i \cup U_i$ in $\mathcal{Q}$. Otherwise, we have $k = f^{-1}(b_i)$ and k is a cascading point on B_i. The choice of k_i and Corollary 3.14 imply $k_i{\downarrow_{\mathcal{P}}} \subseteq k{\downarrow_{\mathcal{P}}}$. Therefore, k can be matched with $f(k_i)$ and k_i can be matched with b_i in $\mathcal{Q}$. All the other properties needed for $\mathcal{Q}$ and Δ are quite obvious.

Case 3.2.2. $r = 1$ (see Figure 19).

Unlike in the previous case, the cascade $k_s, \dots, k_1$ pulls one color from A to C and as such is not redundant in $\mathcal{P}$. The idea is to replace points b_s, k_s with q_s and to shift the cascade after point p. This can be interpreted as replacing cascade $k_s, k_{s-1}, \dots, k_1$ with $p, k_{s-1}, \dots, k_1$.

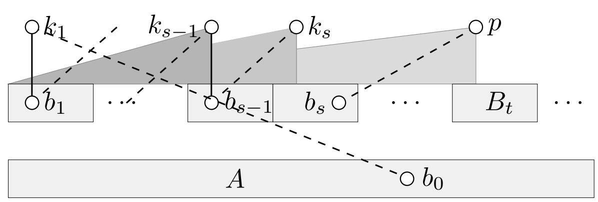


Figure 19. Property (**l**), Case 3.2.2, $r = 1$

We define $\mathcal{Q}$ in which:

(i) Points b_s and k_s are replaced with q_s, played in phase B_{k+1}. (Introducing q_s guarantees that the number of colors used by Γ for $\mathcal{P}$ is the same as the one used by Δ for $\mathcal{Q}$.)

(ii) Cascade $k_{s-1}, \dots, k_1$ is played at the end of the phase C_k (the last phase on the level C, in particular it is played after p which is supposed to get the color from b_{s-1}); $k_i{\downarrow_{\mathcal{Q}}} = K_i{\downarrow_{\mathcal{P}}}$, $i = 1, \dots, s-1$ (downsets of k_i's are extended in $\mathcal{Q}$ so that (**j**) is fulfilled).

(iii) The remaining cascading points in K_i ($i = 1, \dots, s-1$) are made incomparable with b_i, i.e., $k{\downarrow_{\mathcal{Q}}} = k{\downarrow_{\mathcal{P}}} \setminus \{b_i\}, \text{ for } k \in K_i \setminus \{k_i\}, \ i = 1, \dots, s-1$. (This preserves the color of b_i for the final cascade.)

(iv) Points from the set B_i $(i = 1, \ldots, k)$ are made incomparable with $b_0 \in A$, i.e., $b{\downarrow_{\mathcal{Q}}} = b{\downarrow_{\mathcal{P}}} \setminus \{b_0\}$, for $b \in B_1 \cup \ldots \cup B_k$.
(This preserves the color of b_0 for the final cascade.)

The (natural) coloring Δ of $\mathcal{Q}$ is defined as in the previous case, i.e., $\Delta(q_s) = \Gamma(b_s)$, $\Delta(p) = \Gamma(b_{s-1})$, $\Delta(q) = \Gamma(q)$ for $q \notin \{b_s, b_{s-1}\}$.

To prove that $\mathsf{width}\,(B_i \cup U_i) = |B_i|$ for $i = 1, \ldots, s-1$ we again use a matching argument. Since $\mathsf{width}\,(B_i \cup U_i) = |B_i|$ in $\mathcal{P}$ we may construct injective $f : U_i \to B_i$. Again, if there is no $k \in U_i$ such that $f(k) = b_i$ or such k exists and $k > b_i$ in $\mathcal{Q}$ then f witnesses $\mathsf{width}\,(B_i \cup U_i) = |B_i|$ in $\mathcal{Q}$. Otherwise, since k_i has minimal downset among points from K_i dominating b_i we have $k_i{\downarrow_{\mathcal{P}}} \subseteq k{\downarrow_{\mathcal{P}}}$. We can now match k with $f(k_i)$ and k_i with b_i in $\mathcal{Q}$.

From property (**k**) it follows that $\mathsf{width}\,(B \cup C) = |B|$ in $\mathcal{Q}$. To prove that $\mathsf{width}\,(\mathcal{Q}) = \mathsf{width}\,(\mathcal{P})$ it is thus enough to show that $\mathsf{width}\,(A \cup B) = |A|$ in $\mathcal{Q}$. Since $\mathsf{width}\,(A \cup B) = |A|$ in $\mathcal{P}$ (by (**d**)) there is an injection $g : B \to A$ such that $b > g(b)$ in $\mathcal{P}$ for any $b \in B$. If there is no $b \in B$ such that $g(b) = b_0$ or such b exists and $b > b_0$ in $\mathcal{Q}$, we match q_s with $g(b_s)$ and this new matching certifies the width. Otherwise, we have $b > g(b) = b_0$ in $\mathcal{P}$ and $b \not> b_0$ in $\mathcal{Q}$. Note that $B_1, \ldots, B_s$ precedes k_s in $\mathcal{P}$, k_s precedes k_1 in $\mathcal{P}$ and k_1 got its color from b_0. Since b got a new Γ-color and $b > b_0$ we conclude that b had to be played after k_1, hence after $B_1, \ldots, B_s$. From $b_s{\downarrow} \subseteq b{\downarrow}$ we get $b > g(b_s)$ in $\mathcal{Q}$. Now b can be matched with $g(b_s)$ and q_s can be matched with b_0. This new matching witnesses $\mathsf{width}\,(A \cup B) = |A|$ in $\mathcal{Q}$.

Again we leave it to the reader to verify the other properties needed for $\mathcal{Q}$ and Δ. This completes our proof of Property (**l**). $\square$

The following fact is an easy corollary from (**l**).

Corollary 3.17. *Colors from $\Gamma(A)$ are reused only in $B_0 \cup K_1$, colors from $\Gamma(B_i)$ $(1 \leqslant i \leqslant k)$ are reused only in $W_i \cup K_{i+1}$.*

Proof. Fix $i > 1$. By (**j**) the best (with respect to $\prec$) candidates for Γ-color of $k \in K_i$ are points from B_{i-1}. By (**l**) there always exists some Γ-top in B_{i-1} that can be used to color $k \in K_i$, hence, $\Gamma(K_i) \subseteq \Gamma(B_{i-1})$. $\square$

Claim 3.18.

$$b_1 + \ldots + b_i \leqslant b_0 + w_1 + \ldots + w_{i-1}, \qquad i = 1, \ldots, k.$$

Proof. Consider an initial part of the presentation $\mathcal{P}$: let $P^{[0]} = (A, B_0)$, $P^{[i]} = (A, B_0, B_1, \ldots, B_i, C_i)$ for $i \geqslant 1$. Let $F^{[i]}$ be the set of points from A whose Γ-colors are reused by some points in $P^{[i]} \setminus A$ and let $f^{[i]} = \left|F^{[i]}\right|$. From (**f**) and the fact that point $b \in B_i$ is played after the last point of C_{i-1} obtains a new color it follows that $B_i{\downarrow} \subseteq F^{[i-1]}$. On the other hand from $|A| = \mathsf{width}\,(\mathcal{P})$ we get $|B_1| + |B_2| + \ldots + |B_i| \leqslant |B_i{\downarrow}|$. The combination of the two inequalities yields:

$$b_1 + b_2 + \ldots + b_i \leqslant |B_i{\downarrow}| \leqslant f^{[i-1]}, \text{ for } i = 1, \ldots, k. \tag{14}$$

A similar width argument using (**k**) shows $k_j = |K_j| \leqslant |K_j{\downarrow} \cap B_j|$. The fact that points in K_j are not colored with colors from B_j implies $|K_j{\downarrow} \cap B_j| \leqslant |F_j|$. With Corollary 3.17 we thus get

$$k_j \leqslant |K_j{\downarrow} \cap B_j| \leqslant f_j = w_j + k_{j+1}, \; j = 1, \ldots, k.$$

All this is also valid for the presentation of the initial part $P^{[i]}$. Let $b_j^{[i]}$, $w_j^{[i]}$ and $k_j^{[i]}$ be the respective counting variables. Note that for all $j \leqslant i$ we have $b_j^{[i]} = b_j$ (all points from B_j are already introduced) and $w_j^{[i]} = w_j$ (by (**i**) points from W_j

are introduced at the beginning of C_i) and $k_j^{[i]} \leqslant k_j$. Therefore we will omit the notational overhead at the b's and w's.

From above considerations together with $k_{i+1}^{[i]} = 0$ we obtain:

$$k_1^{[i]} \leqslant w_1 + k_2^{[i]} \leqslant w_1 + w_2 + k_3^{[i]} \leqslant \ldots \leqslant w_1 + \ldots + w_i. \tag{15}$$

From Corollary 3.17 it follows that $f^{[i]} = b_0 + k_1^{[i]}$. Put together with (14) and (15) the Claim is proved. □

So far all changes on the order presented by Spoiler were dependent of his original strategy. With the next property we break this scheme. We introduce extra points in level C to simplify forthcoming calculations.

Property (m).

$$\begin{aligned} b_i &= w_i + k_i \quad \text{for} \quad i = 1, \ldots k, \\ w &\leqslant f + b_0. \end{aligned}$$

Proof. Property (**k**) implies $b_i \geqslant k_i + w_i$. Suppose $b_i > w_i + k_i$. Consider $\mathcal{Q}$ which is obtained from $\mathcal{P}$ by adding a new point p which goes into W_i. Point p is played at the beginning of C_i with $p\!\downarrow = A \cup B_1 \cup \ldots \cup B_i$. A natural coloring of $\mathcal{Q}$ obviously needs (at least) as many colors as a natural coloring of $\mathcal{P}$. A straightforward case consideration proves almost all desired properties of $\mathcal{Q}$. We only argue the the width of the poset does not increase by showing that the local width property (**k**) holds in B_i. This follows from the fact that $b_i > k_i + w_i$ in $\mathcal{P}$ and $p > B_i$ in $\mathcal{Q}$.

To prove the second equation we first have an auxiliary claim:

Claim 3.19. $|B| \leqslant f + b_0$.

$$\begin{aligned} |B| = |B_1 \cup \ldots \cup B_{k+1}| + |B_0| &\overset{(\mathbf{d})}{\leqslant} |(B_1 \cup \ldots \cup B_{k+1})\!\downarrow\!| + |B_0| \\ &\overset{(\mathbf{g})}{=} |B_{k+1}\!\downarrow\!| + |B_0| \leqslant |F| + |B_0|. \end{aligned}$$

The last inequality follows from the fact that B_{k+1} may dominate only those points in A whose color is reused in $B \cup C$ (precisely, in $B_0 \cup K_1$, by (**l**)).

Now assume that $w > f + b_0$. As above, we construct a new poset $\mathcal{Q}$ in which the set B_0 contains an additional point. From the claim it follows that $|B| + 1 \leqslant w$ and $\mathsf{width}\,(A \cup B) \leqslant w$ in $\mathcal{Q}$. Repeat until $w \leqslant f + b_0$. □

3.8. **Calculations.** We are ready to state the constraints for the linear program whose solution will prove that the strategy of Spoiler presented in Section 3.4 is indeed optimal.

Theorem 3.20. *If $\mathcal{P}$ is an up-growing semi-order and Γ is a natural coloring of $\mathcal{P}$ using $\mathsf{val}(w)$ colors and satisfying properties* (**a**)–(**m**), *then*

$$(\star) \begin{cases} b_1 + \ldots + b_i \leqslant b_0 + w_1 + \ldots + w_{i-1}, & i = 1, \ldots, k, \\ b_i - 2w_i \leqslant b_{i+1} - w_{i+1}, & i = 1, \ldots, k-1, \\ b_k \leqslant 2w_k, & \\ w - 2b_0 \leqslant b_1 - w_1. & \end{cases}$$

Proof. The first set of inequalities was subject to Claim 3.18.

For the second set of inequalities recall from (**m**) that $b_i = w_i + k_i$ with Corollaries 3.16 and 3.17 we get $b_i = w_i + k_i \leqslant w_i + f_i = 2w_i + k_{i+1} = 2w_i + b_{i+1} - w_{i+1}$ which is just a rearrangement.

The proof of the third is much as previous one: $b_k = w_k + k_k \leqslant w_k + f_k = 2w_k$, where $f_k = w_k$ since C_k is the last group of points on level C.

For the last inequality start with $w - 2b_0 \leqslant f - b_0$ from (**m**), replace f and then k_1 using Corollaries 3.17 and again (**m**). $\square$

Claim 3.21. From the system $(\star)$ of inequalities it follows that

$$\frac{w - b_0}{w} \leqslant \frac{F_{2k+2}}{F_{2k+3}}.$$

Proof. Substituting $\alpha_i = b_i - w_i$ for $i = 1, \ldots, k$, let $\alpha_0 = w - b_0$ and $\alpha_{k+1} = 0$ in $(\star)$ we obtain

$$\begin{cases} b_i \leqslant w - \sum_{j=0}^{i-1} \alpha_j, & i = 1, \ldots, k, \\ 2\alpha_i \leqslant b_i + \alpha_{i+1}, & i = 1, \ldots, k, \\ 2\alpha_0 \leqslant w + \alpha_1. \end{cases}$$

Eliminate b_i from the second set of inequalities by using the first:

$$\alpha_i + \sum_{j=0}^{i} \alpha_j \leqslant w + \alpha_{i+1}, \quad i = 0, \ldots, k. \tag{16}$$

Summation of the inequalities from (16) with weights $F_{2(k-i)+1}$ yields:

$$\sum_{i=0}^{k} \alpha_i F_{2(k-i)+1} + \sum_{i=0}^{k} \sum_{j=0}^{i} \alpha_j F_{2(k-i)+1} \leqslant w \sum_{i=0}^{k} F_{2(k-i)+1} + \sum_{i=0}^{k} \alpha_{i+1} F_{2(k-i)+1}. \tag{17}$$

Recall from (4) that $\sum_{i=0}^{k} F_{2(k-i)+1} = F_{2k+2}$ and hence

$$\sum_{i=0}^{k} \sum_{j=0}^{i} \alpha_j F_{2(k-i)+1} = \sum_{j=0}^{k} \alpha_j \sum_{i=j}^{k} F_{2(k-i)+1} = \sum_{i=0}^{k} \alpha_i F_{2(k-i)+2}.$$

Equation (17) can be now rewritten into

$$\sum_{i=0}^{k} \alpha_i F_{2(k-i)+1} + \sum_{i=0}^{k} \alpha_i F_{2(k-i)+2} \leqslant w F_{2k+2} + \sum_{i=1}^{k+1} \alpha_i F_{2(k-i)+3}.$$

Thus, $\alpha_0 F_{2k+3} = \alpha_0 (F_{2k+1} + F_{2k+2}) \leqslant w F_{2k+2}$. This can be rewritten as:

$$\frac{w - b_0}{w} \leqslant \frac{F_{2k+2}}{F_{2k+3}}.$$

$\square$

Theorem 3.22. *The value of the chain partitioning game on the class of up-growing semi-orders of width w does not exceed $\lfloor \varphi \cdot w \rfloor$.*

Proof. Fix $w \geqslant 0$. Consider a semi-order $\mathcal{P}$ and its natural coloring Γ such that $|\Gamma(\mathcal{P})| = \mathsf{val}(w)$. We may assume that pair $(\mathcal{P}, \Gamma)$ satisfies Properties (**a**)–(**m**). From the definition of B_0 and the fact that no Γ-color is used on C (by 3.9) it follows that

$$\mathsf{val}(w) = \Gamma(\mathcal{P}) = |A| + |B_1| + \ldots + |B_{k+1}| \leqslant w + w - b_0.$$

We know that $(\star)$ holds for $\mathcal{P}$. Therefore we can use Claim 3.21 to get

$$\frac{\mathsf{val}(w)}{w} = 1 + \frac{w - b_0}{w} \leqslant \frac{F_{2k+3} + F_{2k+2}}{F_{2k+3}} \leqslant \frac{F_{2k+4}}{F_{2k+3}} \leqslant \frac{1 + \sqrt{5}}{2}.$$

(The last inequality is due to the fact that the sequence $(\frac{F_{2k+2}}{F_{2k+1}})_{k \geqslant 0}$ is monotone increasing with limit $\varphi = \frac{1+\sqrt{5}}{2}$). This, together with the fact that $\mathsf{val}(w)$ is an integer, completes the proof. $\square$

4. Conclusion

The big problem in the field of on-line chain partitioning remains to lower the gap between upper $\frac{5^w-1}{4}$ and lower bound $\binom{w+1}{2}$ in the unrestricted setting. With the results of this paper, however, we have again seen that considering variants and restricted versions of the general problem can lead to interesting structures and beautiful mathematics. We feel that together with the restrictions to special classes of orders, two types of restriction which reduce the power of Spoiler are interesting:

- The up-growing case.
- The case where Spoiler has to present the order with a geometric representation which proves the membership of the order in a given class.

Below is a table of some related results and open problems. The columns **U** and **R** of the table indicate whether Spoiler has to play up-growing and with a geometric representation, respectively. In particular it would be very interesting to answer the following questions:

Problem 1. What is the value of the on-line chain partitioning game for semi-orders with geometrical representation? It is easy to see that in this case $\frac{3}{2}w \leqslant \mathsf{val}(w) \leqslant 2w-1$. Moreover, any greedy on-line algorithm may be forced to use $2w-1$ chains. There are two variants of this problem, either Spoiler presents a proper or a unit interval representation. Both problems are open, it is likely that the value of these two games is different.

Problem 2. What is the value of the on-line chain partitioning game of 3-dimensional orders with geometrical representation? In this case (by [6]) we have $\binom{w+1}{2} \leqslant \mathsf{val}(w) \leqslant \binom{w+1}{2}^2$.

	class	**U**	**R**	**value**	**remarks**
1	all orders			?	$\binom{w+1}{2} \leqslant ? \leqslant \frac{5^w-1}{4}$, [10], [4]
2	all orders	+		$\binom{w+1}{2}$	[3]
3	interval orders			$3w-2$	[7]
4	interval orders		+	$3w-2$	[7], [2]
5	interval orders	+		$2w-1$	[1]
6	interval orders	+	+	w	obvious
7	semi-orders			$2w-1$	cf. introduction
8	semi-orders		+	?	$\frac{3}{2}w \leqslant ? \leqslant 2w-1$
9	semi-orders	+		$\lfloor \frac{1+\sqrt{5}}{2} w \rfloor$	this paper
10	semi-orders	+	+	w	from line 6
11	2-dimensional			?	$\binom{w+1}{2} \leqslant ? \leqslant \frac{5^w-1}{4}$
12	2-dimensional		+	$\binom{w+1}{2}$	[9], [10]
13	2-dimensional	+		$\binom{w+1}{2}$	from lines 2 and 14
14	2-dimensional	+	+	$\binom{w+1}{2}$	[3] and this paper
15	d-dimensional		+	?	$\binom{w+1}{2} \leqslant ? \leqslant \binom{w+1}{2}^{d-1}$, [6]

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